5. Intégration des fonctions du type : $\int \frac{\alpha x+\beta}{ax^2+bx+c}\,dx$

On procède par décomposition en éléments simples sur $\mathbb{R}$.

Trois cas différents se présentent :

$\boxed{\text{a) } 1^{er} \text{ cas}}$: le dénominateur admet deux racines réelles distinctes : $x_1 \neq x_2$

$$\int \frac{\alpha x+\beta}{ax^2+bx+c}\,dx = \int \frac{A}{x-x_1} + \frac{B}{x-x_2}\,dx = A\ln|x-x_1| + B\ln|x-x_2|$$

1. $I_1 = \int \frac{5x-3}{x^2-2x-3}\,dx$

$1^{ère}$ étape : Les racines du dénominateur sont $x_1 = 3$ et $x_2 = -1$ donc on peut l'écrire sous la forme : $x^2-2x-3 = (x-3)(x-(-1)) = (x-3)(x+1)$

$2^{ème}$ étape : Décomposition en éléments simples sur $\mathbb{R}$ de $\frac{5x-3}{x^2-2x-3}$

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{Ax+A+Bx-3B}{(x-3)(x+1)} \Rightarrow \frac{5x-3}{x^2-2x-3} = \frac{(A+B)x+A-3B}{(x-3)(x+1)}$$

Le dénominateur étant le même, alors :

$$5x-3 = (A+B)x + A - 3B \Rightarrow \begin{cases} A+B=5 \\ A-3B=-3 \end{cases} \quad \text{d'où} \quad A = 3 \quad \text{et } B = 2$$

Ainsi $\frac{5x-3}{x^2-2x-3} = \frac{3}{x-3} + \frac{2}{x+1}$

$3^{ème}$ étape : l'intégration :

$$I_1 = \int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{3}{x-3} + \frac{2}{x+1}\,dx = \int \frac{3}{x-3}\,dx + \int \frac{2}{x+1}\,dx$$

$$I_1 = 3\int \frac{1}{x-3}\,dx + 2\int \frac{1}{x+1}\,dx \Rightarrow I_1 = 3\ln|x-3| + 2\ln|x+1| + C$$

2) $I_2 = \int \frac{x+4}{x^2+5x-6}dx$

$1^{ère}$ étape: les racines de x^2+5x-6 sont $x_1=1$ et $x_2=-6$ d'où
$x^2+5x-6=(x-1)(x+6)$

$2^{ème}$ étape: $\frac{x+4}{x^2+5x-6} = \frac{x+4}{(x-1)(x+6)} = \frac{A}{x-1} + \frac{B}{x+6} = \frac{A(x+6)+B(x-1)}{(x-1)(x+6)}$

$$\frac{x+4}{x^2+5x-6} = \frac{(A+B)x+6A-B}{(x-1)(x+6)} \Rightarrow \begin{cases} A+B=1 \\ 6A-B=4 \end{cases} \Rightarrow \begin{cases} A=\frac{5}{7} \\ B=\frac{2}{7} \end{cases}$$

$3^{ème}$ étape: $\frac{x+4}{x^2+5x-6} = \frac{5/7}{x-1} + \frac{2/7}{x+6}$ d'où

$$I_2 = \int \frac{x+4}{x^2+5x-6}dx = \int \frac{5/7}{x-1} + \frac{2/7}{x+6}dx = \frac{5}{7}\int \frac{1}{x-1}dx + \frac{2}{7}\int \frac{1}{x+6}dx$$

$$I_2 = \frac{5}{7}\ln|x-1| + \frac{2}{7}\ln|x+6| + C$$

3) $I_3 = \int \frac{2\,dx}{x^2-7x+12} = \int \frac{2}{(x-3)(x-4)}dx$

$$\frac{2}{x^2-7x+12} = \frac{0.x+2}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4} = \frac{(A+B)x-4A-3B}{(x-3)(x-4)}$$

$$\begin{cases} A+B=0 \\ -4A-3B=2 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=2 \end{cases}$$ d'où $I_3 = \int \frac{-2}{x-3} + \frac{2}{x-4}dx$

$$I_3 = -2\ln|x-3| + 2\ln|x-4| = 2\ln\left|\frac{x-4}{x-3}\right| + C$$

4) $I_4 = \int \frac{2x-7}{3x^2-11x-4}$. Les racine de $3x^2-11x-4$ sont $x_1=4$, $x_2=-\frac{1}{3}$

d'où $3x^2-11x-4 = 3(x-4)(x+\frac{1}{3})$

$$\frac{2x-7}{3x^2-11x-4} = \frac{2x-7}{3(x-4)(x+\frac{1}{3})} = \frac{1}{3}\left[\frac{A}{x-4} + \frac{B}{x+\frac{1}{3}}\right] = \frac{1}{3}\cdot\frac{(A+B)x+\frac{1}{3}A-4B}{(x-4)(x+\frac{1}{3})}$$

$$\begin{cases} A+B=2 \\ \frac{1}{3}A-4B=7 \end{cases} \Rightarrow \begin{cases} A=45/13 \\ B=-19/13 \end{cases}$$ d'où $\frac{2x-7}{3x^2-11x-4} = \frac{1}{3}\cdot\left[\frac{45/13}{x-4} - \frac{19/13}{x+\frac{1}{3}}\right]$

$$I_4 = \int \frac{2x-7}{3x^2-11x-4}dx = \int \frac{1}{3}\left[\frac{45/13}{x-4} - \frac{19/13}{x+\frac{1}{3}}\right]dx = \frac{15}{13}\int \frac{dx}{x-4} - \frac{19}{39}\int \frac{dx}{x+\frac{1}{3}}$$

$$I_4 = \frac{15}{13}\ln|x-4| - \frac{19}{39}\ln\left|x+\frac{1}{3}\right| + C$$

b) $2^{\text{ème}}$ cas : le dénominateur admet une racine réelle double x_0.

$$\int \frac{\alpha x+\beta}{ax^2+bx+c}dx = \int \frac{A}{x-x_0} + \frac{B}{(x-x_0)^2}dx = A\ln|x-x_0| - \frac{B}{x-x_0}$$

$$I_1 = \int \frac{6x+7}{x^2+4x+4}dx = \int \frac{6x+7}{(x+2)^2}dx.$$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)}{(x+2)^2} + \frac{B}{(x+2)^2} = \frac{Ax+2A+B}{(x+2)^2}$$

$$\begin{cases} A=6 \\ 2A+B=7 \end{cases} \Rightarrow \begin{cases} A=6 \\ B=-5 \end{cases}$$

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} + \frac{-5}{(x+2)^2} \Rightarrow I_1 = \int \frac{6}{x+2} + \frac{-5}{(x+2)^2}dx = 6\int\frac{dx}{x+2} - 5\int\frac{dx}{(x+2)^2}$$

$$I_1 = 6\ln|x+2| - 5\int (x+2)^{-2}dx = 6\ln|x+2| - 5.\frac{1}{-2+1}(x+2)^{-2+1}$$

$$I_1 = 6\ln|x+2| + \frac{5}{x+2} + C$$

$$I_2 = \int \frac{2x+2}{x^2-2x+1}dx = \int \frac{2x+2}{(x-1)^2}dx$$

$$\frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)}{(x-1)^2} + \frac{B}{(x-1)^2} = \frac{Ax-A+B}{(x-1)^2}$$

$$\begin{cases} A=2 \\ -A+B=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=4 \end{cases}$$

$$\frac{2x+2}{(x-1)^2} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$$

$$I_2 = \int \frac{2x+2}{x^2-2x+1}dx = \int \frac{2}{x-1} + \frac{4}{(x-1)^2}dx = 2\int\frac{dx}{x-1} + 4\int\frac{dx}{(x-1)^2}$$

$$I_2 = 2\ln|x-1| + 4\int(x-1)^{-2}dx = 2\ln|x-1| + 4.\frac{1}{-2+1}(x-1)^{-2+1}$$

$$I_2 = 2\ln|x-1| - \frac{4}{x-1} + C$$

c) $3^{ème}$ cas : le dénominateur n'admet pas de solutions réelles:

$$I = \int \frac{\alpha x + \beta}{ax^2 + bx + c}\, dx$$

1er sous-cas : $\alpha = 0 \Rightarrow I = \beta \int \frac{dx}{ax^2 + bx + c}$

On utilise alors le changement de variable : $t = 2ax + b$

1) $I_1 = \int \frac{dx}{2x^2 - 5x + 7}$ on pose $t = 2.2x + (-5) \Rightarrow t = 4x - 5$

ainsi $dt = 4dx \Rightarrow dx = \frac{1}{4} dt$

de plus $t = 4x - 5 \Rightarrow x = \frac{t+5}{4}$

d'où $I_1 = \int \frac{dx}{2x^2 - 5x + 7} = \int \frac{\frac{1}{4} dt}{2\times\left(\frac{t+5}{4}\right)^2 - 5\times\left(\frac{t+5}{4}\right) + 7}$

$$I_1 = \frac{1}{4} \int \frac{dt}{2\left(\frac{t^2 + 10t + 25}{16}\right) + \frac{-5t - 25}{4} + 7} = \frac{1}{4} \int \frac{dt}{\frac{t^2 + 10t + 25 - 10t + 50 + 56}{8}}$$

$$I_1 = \frac{1}{4} \int \frac{8dt}{t^2 + 31} = 2 \int \frac{dt}{31\left(\frac{t^2}{31} + 1\right)} = \frac{2}{31} \int \frac{dt}{\left(\frac{t}{\sqrt{31}}\right)^2 + 1}$$

on pose $\frac{t}{\sqrt{31}} = z \Rightarrow \frac{dt}{\sqrt{31}} = dz \Rightarrow dt = \sqrt{31}\, dz$ d'où

$$I_1 = \frac{2}{31} \cdot \int \frac{\sqrt{31}\, dz}{z^2 + 1} = \frac{2.\sqrt{31}}{31} \int \frac{dz}{z^2 + 1} = \frac{2}{\sqrt{31}} \arctan z$$

$$I_1 = \frac{2}{\sqrt{31}} \arctan\left(\frac{t}{\sqrt{31}}\right) = \frac{2}{\sqrt{31}} \arctan\left(\frac{4x - 5}{\sqrt{31}}\right) + C$$

2) $I_2 = \int \frac{dx}{3x^2 - x + 4}$ on pose $2.3x + (-1) = t \Rightarrow t = 6x - 1 \Rightarrow x = \frac{t+1}{6}$

et $dx = \frac{1}{6} dt$

$$I_2 = \int \frac{dx}{3x^2 - x + 4} = \int \frac{\frac{1}{6} dt}{3\cdot\left(\frac{t+1}{6}\right)^2 - \left(\frac{t+1}{6}\right) + 4} = \frac{1}{6} \int \frac{dt}{\frac{t^2+2t+1}{12} - \frac{2(t+1)}{12} + \frac{48}{12}}$$

$$I_2 = \frac{1}{6} \int \frac{12 dt}{t^2 + 47} = 2 \int \frac{dt}{47\left(\frac{1}{47}t^2 + 1\right)} = \frac{2}{47} \int \frac{dt}{\left(\frac{1}{\sqrt{47}}t\right)^2 + 1}$$

on pose $\frac{1}{\sqrt{47}} t = z \Rightarrow \frac{dt}{\sqrt{47}} = dz \Rightarrow dt = \sqrt{47}\, dz$

$$I_2 = \frac{2}{47} \int \frac{\sqrt{47}\, dz}{z^2 + 1} = \frac{2}{\sqrt{47}} \arctan z = \frac{2}{\sqrt{47}} \arctan \frac{t}{\sqrt{47}}$$

$$I_2 = \frac{2}{\sqrt{47}} \arctan\left(\frac{6x - 1}{\sqrt{47}}\right) + C$$

3) L'intégrale précédente peut se calculer d'une autre manière. Au lieu de faire le changement de variable $t = 6x - 1$, on essaie d'écrire le dénominateur sous la forme d'un carré complet $(x + \square)^2 = x^2 + 2\square x + \square^2$

$$I_2 = \int \frac{dx}{3x^2 - x + 4} = \frac{1}{3} \int \frac{dx}{x^2 - \frac{1}{3}x + \frac{4}{3}} = \frac{1}{3} \int \frac{dx}{x^2 - 2 \times \frac{1}{2} \times \frac{1}{3} x + \frac{4}{3}} = \frac{1}{3} \int \frac{dx}{x^2 - 2 . \frac{1}{6} x + \frac{4}{3}}$$

$$I_2 = \frac{1}{3} \int \frac{dx}{x^2 - 2 . \boxed{\frac{1}{6}} x + \frac{1}{36} - \frac{1}{36} + \frac{4}{3}} = \frac{1}{3} \int \frac{dx}{\left(x^2 - 2 . \frac{1}{6} x + \frac{1}{36}\right) + \left(-\frac{1}{36} + \frac{4}{3}\right)}$$

$\left(\frac{1}{6}\right)^2 = 36$

$$I_2 = \frac{1}{3} \int \frac{dx}{\left(x - \frac{1}{6}\right)^2 + \frac{47}{36}}$$ on pose $x - \frac{1}{6} = t \Rightarrow dx = dt$ $\quad I_2 = \frac{1}{3} \int \frac{dt}{t^2 + \frac{47}{36}}$

$$I_2 = \frac{1}{3} \int \frac{dt}{\frac{47}{36}\left(\frac{36}{47} t^2 + 1\right)} = \frac{1}{3} \cdot \frac{36}{47} \int \frac{dt}{\left(\frac{6t}{\sqrt{47}}\right)^2 + 1} = \frac{12}{47} \int \frac{dt}{\left(\frac{6t}{\sqrt{47}}\right)^2 + 1}$$

on pose $\frac{6t}{\sqrt{47}} = z \Rightarrow \frac{6}{\sqrt{47}} dt = dz \Rightarrow dt = \frac{\sqrt{47}}{6} dz$ d'où

$$I_2 = \frac{12}{47} \int \frac{\frac{\sqrt{47}}{6} dz}{z^2 + 1} = \frac{12}{47} \times \frac{\sqrt{47}}{6} \int \frac{dz}{z^2 + 1} = \frac{2}{\sqrt{47}} \arctan z = \frac{2}{\sqrt{47}} \arctan\left(\frac{6}{\sqrt{47}} t\right)$$

$$I_2 = \frac{2}{\sqrt{47}} \arctan\left(\frac{6}{\sqrt{47}}\left(x - \frac{1}{6}\right)\right) = \frac{2}{\sqrt{47}} \arctan\left(\frac{6x - 1}{\sqrt{47}}\right) + C.$$

$2^{\text{ème}}$ sous-cas : $\alpha \neq 0$: $I = \displaystyle\int \frac{\alpha x + \beta}{ax^2+bx+c}\,dx$

1) $I_1 = \displaystyle\int \frac{x-1}{x^2-x+1}\,dx$

$1^{\text{ère}}$ étape : on fait apparaître la dérivée du dénominateur $(x^2-x+1)' = 2x-1$ au numérateur : $A(2x-1)+B = x-1 \Rightarrow 2Ax - A + B = x-1 \Rightarrow \begin{cases} 2A = 1 \\ -A+B = -1 \end{cases}$

d'où $A = 1/2$ et $B = -\frac{1}{2}$ d'où

$$I_1 = \int \frac{1/2(2x-1) - 1/2}{x^2-x+1}\,dx = \underbrace{\frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx}_{L1} - \underbrace{\frac{1}{2}\int \frac{dx}{x^2-x+1}}_{L_2}$$

$$L_1 = \frac{1}{2}\int \frac{\boxed{2x-1}^{\,f'}}{\underset{f}{\boxed{x^2-x+1}}}\,dx = \frac{1}{2}\ln|x^2-x+1|$$

$$L_2 = -\frac{1}{2}\int \frac{dx}{x^2-x+1}$$ on pose $\left.\begin{array}{l} 2x-1 = t \\ x = \frac{t+1}{2} \end{array}\right\} \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$

$$L_2 = -\frac{1}{2}\int \frac{\frac{1}{2}dt}{\left(\frac{t+1}{2}\right)^2 - \left(\frac{t+1}{2}\right) + 1} = -\frac{1}{4}\int \frac{dt}{\frac{t^2+2t+1-2t-2+4}{4}} = -\frac{1}{4}\int \frac{4\,dt}{t^2+3}$$

$$L_2 = -\int \frac{dt}{t^2+3} = -\int \frac{dt}{3\left(\frac{t^2}{3}+1\right)} = -\frac{1}{3}\int \frac{dt}{\left(\frac{t}{\sqrt{3}}\right)^2+1}$$

on pose $\frac{t}{\sqrt{3}} = z \Rightarrow \frac{dt}{\sqrt{3}} = dz \Rightarrow dt = \sqrt{3}\,dz$

$$L_2 = -\frac{1}{3}\int \frac{\sqrt{3}\,dz}{z^2+1} = -\frac{\sqrt{3}}{3}\arctan z = -\frac{1}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$$

d'où

$$I_2 = \frac{1}{2}\ln|x^2-x+1| - \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$$

2) $I_2 = \int \frac{3x-2}{x^2-4x+5}\,dx$, nous avons $(x^2-4x+5)' = 2x-4$

d'où $A(2x-4)+B = 3x-2 \Rightarrow 2Ax - 4A + B = 3x - 2 \Rightarrow \begin{cases} 2A = 3 \\ -4A+B = -2 \end{cases} \Rightarrow \begin{cases} A = 3/2 \\ B = 4 \end{cases}$

$$I_2 = \int \frac{3/2(2x-4)+4}{x^2-4x+5}\,dx = \underbrace{\frac{3}{2}\int \frac{2x-4}{x^2-4x+5}\,dx}_{L_1} + \underbrace{4\int \frac{dx}{x^2-4x+5}}_{L_2}$$

$$L_1 = \frac{3}{2}\int \frac{2x-4}{x^2-4x+5}\,dx = \frac{3}{2}\ln|x^2-4x+5|$$

$L_2 = 4\int \frac{dx}{x^2-4x+5}$ on pose $2x-4 = t \Rightarrow x = \frac{t+4}{2}$ et $dx = \frac{1}{2}dt$

$$L_2 = 4\cdot\int \frac{\frac{1}{2}dt}{\left(\frac{t+4}{2}\right)^2 - 4\cdot\left(\frac{t+4}{2}\right)+5} = 2\int \frac{dt}{\frac{t^2+8t+16-8t-32+20}{4}}$$

$$L_2 = 2\int \frac{4\,dt}{t^2+4} = 8\int \frac{dt}{t^2+4} = 8\int \frac{dt}{4(\frac{1}{4}t^2+1)} = \frac{8}{4}\int \frac{dt}{\left(\frac{t}{2}\right)^2+1} = 2\int \frac{dt}{\left(\frac{t}{2}\right)^2+1}$$

on pose : $\frac{t}{2} = z \Rightarrow \frac{dt}{2} = dz$ et $dt = 2dz$

$$L_2 = 2\int \frac{2dz}{z^2+1} = 4\int \frac{dz}{z^2+1} = 4\arctan z$$

$$L_2 = 4\arctan\frac{t}{2} = 4\arctan\frac{2x-4}{2} = 4\arctan(x-2)$$

Finalement :

$$I_2 = \frac{3}{2}\ln|x^2-4x+5| + 4\arctan(x-2) + c.$$

3) $I_3 = \int \frac{x}{x^2-7x+13}dx$ Nous avons $(x^2-7x+13)' = 2x-7$

d'où $A(2x-7)+B = x \Rightarrow 2Ax-7A+B = x$ d'où

$$\begin{cases} 2A = 1 \\ -7A+B = 0 \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ B = +7/2 \end{cases}$$

$$I_3 = \int \frac{\frac{1}{2}(2x-7)+\frac{7}{2}}{x^2-7x+13}dx = \underbrace{\frac{1}{2}\int \frac{2x-7}{x^2-7x+13}dx}_{L_1} + \underbrace{\frac{7}{2}\int \frac{dx}{x^2-7x+13}}_{L_2}$$

$$L_1 = \frac{1}{2}\int \frac{2x-7}{x^2-7x+13}dx = \frac{1}{2}\ln|x^2-7x+13|$$

$L_2 = \frac{7}{2}\int \frac{dx}{x^2-7x+13}$ ici, on va mettre le dénominateur sous la forme d'un carré complet :

$$L_2 = \frac{7}{2}\int \frac{dx}{x^2-2\times\frac{1}{2}7x+13} = \frac{7}{2}\int \frac{dx}{x^2-2.\frac{7}{2}x+13} = \frac{7}{2}\int \frac{dx}{x^2-2.\frac{7}{2}x+\frac{49}{4}-\frac{49}{4}+13}$$

$$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

$L_2 = \frac{7}{2}\int \frac{dx}{(x^2-2.\frac{7}{2}x+\frac{49}{4})+(-\frac{49}{4}+13)} = \frac{7}{2}\int \frac{dx}{(x-\frac{7}{2})^2+\frac{3}{4}}$ on pose $x-\frac{7}{2} = t$ et $dx = dt$

$$L_2 = \frac{7}{2}\int \frac{dt}{t^2+\frac{3}{4}} = \frac{7}{2}\int \frac{dt}{\frac{3}{4}(\frac{4}{3}t^2+1)} = \frac{7}{2}\times\frac{4}{3}\int \frac{dt}{(\frac{2}{\sqrt{3}}t)^2+1}$$

$L_2 = \frac{14}{3}\int \frac{dt}{(\frac{2}{\sqrt{3}}t)^2+1}$ on pose $\frac{2}{\sqrt{3}}t = z \Rightarrow dz = \frac{2}{\sqrt{3}}dt \Rightarrow dt = \frac{\sqrt{3}}{2}dz$

$$L_2 = \frac{14}{3}\int \frac{\frac{\sqrt{3}}{2}dz}{z^2+1} = \frac{14}{3}.\frac{\sqrt{3}}{2}\int \frac{dz}{z^2+1} = \frac{7}{\sqrt{3}}\arctan z = \frac{7}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}t$$

$$L_2 = \frac{7}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x-\frac{7}{2}\right)\right] = \frac{7}{\sqrt{3}}\arctan\left(\frac{2x-7}{\sqrt{3}}\right)$$

Finalement :

$$I_2 = L_1+L_2 = \frac{1}{2}\ln|x^2-7x+13| + \frac{7}{\sqrt{3}}\arctan\left(\frac{2x-7}{\sqrt{3}}\right) + c$$