Exercices supplémentaires sur l'intégration

1) $I = \int \sin(\ln x)\,dx$. On utilise l'intégration par parties

$$I = \int \underset{\uparrow v'}{1} \cdot \underset{\uparrow u}{\sin(\ln x)}\,dx = u.v - \int v.u'\,dx$$

on pose $v'(x) = 1 \Rightarrow v(x) = x$

$$u(x) = \sin(\ln x) \Rightarrow u'(x) = \frac{1}{x} \cdot \cos(\ln x)$$

$$I = x.\sin(\ln x) - \int x.\frac{1}{x}\cos(\ln x)\,dx$$

$$I = x.\sin(\ln x) - \underbrace{\int \cos(\ln x)\,dx}_{J}$$

Pour calculer J, on utilise l'intégration par parties encore une fois:

$$J = \int \underset{\uparrow v'}{1} . \underset{\uparrow u}{\cos(\ln x)}\,dx = u.v - \int v.u'\,dx$$

$$v'(x) = 1 \Rightarrow v(x) = x$$

$$u(x) = \cos(\ln x) \Rightarrow u'(x) = -\frac{1}{x}.\sin(\ln x)$$

$$J = x.\cos(\ln x) - \int x.\left(-\frac{1}{x}\sin(\ln x)\right)dx$$

$$J = x.\cos(\ln x) + \int \sin(\ln x)\,dx$$

D'où $I = x\sin(\ln x) - \left[x.\cos(\ln x) + \int \sin(\ln x)\,dx\right]$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x.\cos\ln(x) - \int \sin(\ln x)\,dx$$

$$2\int \sin(\ln x)\,dx = x\left(\sin(\ln x) - \cos(\ln x)\right)$$

Finalement : $\int \sin(\ln x)\,dx = \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right]$

2) $I = \int x . \ln \frac{1-x}{1+x} dx$ Intégration par parties

(x ↑ v', $\ln\frac{1-x}{1+x}$ ↑ u)

$v'(x) = x \quad \Rightarrow v(x) = \frac{1}{2}x^2$

$u(x) = \ln \frac{1-x}{1+x} \quad \Rightarrow u'(x) = \frac{\frac{-1(1+x)-1.(1-x)}{(1+x)^2}}{\frac{1-x}{1+x}} = \frac{-2}{(1+x)(1-x)}$

$$I = \frac{1}{2}x^2 . \ln \frac{1-x}{1+x} - \int \frac{1}{2}x^2 . \frac{-2}{(1+x)(1-x)} dx$$

$$I = \frac{1}{2}x^2 . \ln \frac{1-x}{1+x} - \int \frac{-x^2}{1-x^2} dx$$

$$I = \frac{1}{2}x^2 . \ln \frac{1-x}{1+x} - \int \frac{-x^2 +1-1}{1-x^2} dx = \frac{1}{2}x^2 \ln \frac{1-x}{1+x} - \int 1 - \frac{1}{1-x^2} dx$$

$$I = \frac{1}{2}x^2 . \ln \frac{1-x}{1+x} - x + \int \frac{dx}{1-x^2} = \frac{1}{2}x^2 . \ln \frac{1-x}{1+x} - x + \int \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x} dx$$

$$I = \frac{1}{2}x^2 . \ln \frac{1-x}{1+x} - x - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)$$

$$I = \frac{1}{2}x^2 . \ln \frac{1-x}{1+x} - x - \frac{1}{2}\ln \frac{1-x}{1+x}$$

$$I = \frac{1}{2}(x^2 - 1) . \ln \frac{1-x}{1+x} - x$$

3) $I = \int \frac{\ln 2x}{\ln 4x} \frac{dx}{x} = \int \frac{\ln 2x}{\ln(2x \times 2x)} \cdot \frac{dx}{x} = \int \frac{\ln 2x}{\ln 2 + \ln 2x} \cdot \frac{dx}{x}$

on pose $\ln 2x = t \Rightarrow \frac{2}{2x} dx = dt$ d'où $\frac{dx}{x} = dt$

Ainsi, $I = \int \frac{t}{\ln 2 + t} \cdot dt = \int \frac{t + \ln 2 - \ln 2}{t + \ln 2} dt = \int 1 - \frac{\ln 2}{t + \ln 2} dt$

$I = \int dt - \int \frac{\ln 2}{t + \ln 2} dt = t - \ln 2 \int \frac{dt}{t + \ln 2}$

$I = t - \ln 2 \cdot \ln|t + \ln 2|$

$I = \ln 2x - \ln 2 \cdot \ln|\ln 2x + \ln 2|$

$I = \ln 2x - \ln 2 \cdot \ln|\ln 4x|$

4) $I = \int \frac{\sin^3 x \, dx}{\sqrt{\cos x}} = \int \frac{\sin^2 x \cdot \sin x}{\sqrt{\cos x}} dx = \int \frac{(1 - \cos^2 x) \cdot \sin x}{\sqrt{\cos x}} dx$

on pose $\sqrt{\cos x} = t \Rightarrow \frac{-\sin x}{2\sqrt{\cos x}} dx = dt$

$\Rightarrow \frac{\sin x}{\sqrt{\cos x}} dx = -2dt$ d'où

$I = \int (1 - t^4) \cdot (-2\, dt) = 2 \int t^4 - 1 \, dt = 2 \cdot \left(\frac{1}{5} t^5 - t\right)$

$I = \frac{2}{5} t^5 - 2t \Rightarrow I = \frac{2}{5} \left(\sqrt{\cos x}\right)^5 - 2\sqrt{\cos x}$

$I = \frac{2}{5} [\cos^2 x - 5] \sqrt{\cos x}$

5) $I = \int \frac{x}{x^4 - 4x^2 + 3} dx$. on pose $t = x^2 \Rightarrow dt = 2x\,dx$

d'où $x\,dx = \frac{1}{2} dt$

$$I = \frac{1}{2}\int \frac{dt}{t^2 - 4t + 3} = \frac{1}{2}\int \frac{dt}{(t-1)(t-3)} = \frac{1}{2}\int \frac{A}{t-1} + \frac{B}{t-3}\, dt$$

$$I = \frac{1}{2}\int \frac{-\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t-3}\, dt = \frac{1}{2}\left[\frac{1}{2}\ln|t-3| - \frac{1}{2}\ln|t-1|\right]$$

$I = \frac{1}{4}\ln\left|\frac{t-3}{t-1}\right|$. Finalement $I = \frac{1}{4}\ln\left|\frac{x^2-3}{x^2-1}\right|$.

6) $I = \int \frac{\cos x}{\sin^2 x - 6\sin x + 12} dx$. on pose $t = \sin x \Rightarrow dt = \cos x\, dx$

$$I = \int \frac{dt}{t^2 - 6t + 12} = \int \frac{dt}{t^2 - 2\times 3t + 12} = \int \frac{dt}{t^2 - 2\times 3t + 3^2 - 3^2 + 12}$$

n'a pas de racines réelles

$$I = \int \frac{dt}{(t^2 - 2\times 3\times t + 3^2) - 9 + 12} = \int \frac{dt}{(t-3)^2 + 3}$$

$I = \frac{1}{3}\int \frac{dt}{\frac{(t-3)^2}{3} + 1} = \frac{1}{3}\int \frac{dt}{\left(\frac{t-3}{\sqrt{3}}\right)^2 + 1}$ on pose $\frac{t-3}{\sqrt{3}} = z$ d'où

$\frac{1}{\sqrt{3}} dt = dz \Rightarrow dt = \sqrt{3}\, dz$

$$I = \frac{1}{3}\int \frac{\sqrt{3}\, dz}{z^2 + 1} = \frac{1}{\sqrt{3}}\int \frac{dz}{z^2+1} = \frac{1}{\sqrt{3}} \arctan z$$

$$I = \frac{1}{\sqrt{3}} \arctan\left(\frac{t-3}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} \arctan\left(\frac{\sin(x) - 3}{\sqrt{3}}\right)$$

7) $I = \int \frac{\ln x \, dx}{x\sqrt{1-4\ln x - \ln^2 x}}$ on pose $\ln x = t \Rightarrow \frac{1}{x}dx = dt$ d'où

$$I = \int \frac{t}{\sqrt{1-4t-t^2}}dt = -\frac{1}{2}\int \frac{-2t}{\sqrt{1-4t-t^2}}dt$$

$$I = -\frac{1}{2}\int \frac{-2t-4+4}{\sqrt{1-4t-t^2}}dt = -\frac{1}{2}\int \frac{-2t-4}{\sqrt{1-4t-t^2}}dt - \frac{1}{2}\int \frac{4 \, dt}{\sqrt{1-4t-t^2}}$$

$$I = \underbrace{\int \frac{-2t-4}{2\sqrt{1-4t-t^2}}}_{\frac{f'}{2\sqrt{f}}}dt - 2\underbrace{\int \frac{dt}{\sqrt{1-4t-t^2}}}_{J} = -\sqrt{1-4t-t^2} - 2J$$

$$J = \int \frac{dt}{\sqrt{1-4t-t^2}} = \int \frac{dt}{\sqrt{1-(t^2+4t)}} = \int \frac{dt}{\sqrt{1-(t^2+2\times 2t)}} = \int \frac{dt}{\sqrt{1-(t^2+2\times 2t+2^2-2^2)}}$$

$$J = \int \frac{dt}{\sqrt{1-[(t+2)^2-4]}} = \int \frac{dt}{\sqrt{1+4-(t+2)^2}} = \int \frac{dt}{\sqrt{5-(t+2)^2}}$$

$$J = \int \frac{dt}{\sqrt{5}\sqrt{1-\frac{(t+2)^2}{5}}} = \frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{1-\left(\frac{t+2}{\sqrt{5}}\right)^2}}$$ on pose $\frac{t+2}{\sqrt{5}} = z$ d'où

$\frac{1}{\sqrt{5}}dt = dz$, ainsi $J = \int \frac{dz}{\sqrt{1-z^2}} = \arcsin z = \arcsin \frac{t+2}{\sqrt{5}}$

$= \arcsin\left(\frac{\ln x + 2}{\sqrt{5}}\right)$

d'où

$$I = -\sqrt{1-4\ln x - \ln^2 x} - 2\arcsin\left(\frac{\ln x + 2}{\sqrt{5}}\right)$$

8) $I = \int \frac{\sin x}{(1-\cos x)^3} dx$. Vous pouvez très bien utiliser les règles de Bioche mais voici un méthode plus rapide :

on pose $t = -\cos x \Rightarrow dt = -(-\sin x)\, dx$

$dt = \sin x\, dx$ d'où

$$I = \int \frac{dt}{(1+t)^3} = \int 1 \cdot (t+1)^{-3} dt$$ de la forme $\int f'(x) f(x)^n dx = \frac{1}{n+1} f^{n+1}$

$$I = \frac{1}{-3+1} \cdot (t+1)^{-3+1} = -\frac{1}{2}(t+1)^{-2} = \frac{-1}{2(t+1)^2}$$ d'où

$$I = \frac{-1}{2(1-\cos x)^2}$$

9) $I = \int \frac{\cos x}{\sin^2 x - 6\sin x + 5} dx$ même remarque que l'exemple 8

on pose $t = \sin x \Rightarrow dt = \cos x\, dx$

$$I = \int \frac{dt}{t^2 - 6t + 5} = \int \frac{dt}{(t-1)(t-5)} = \int \frac{A}{t-1} + \frac{B}{t-5} dt$$

$$I = \int \frac{-\frac{1}{4}}{t-1} + \frac{\frac{1}{4}}{t-5} dt = -\frac{1}{4}\ln|t-1| + \frac{1}{4}\ln|t-5|$$

$$I = \frac{1}{4}\ln\left|\frac{t-5}{t-1}\right| = \frac{1}{4}\ln\left|\frac{-(5-t)}{-(1-t)}\right| = \frac{1}{4}\ln\left|\frac{5-t}{1-t}\right|$$

$I = \frac{1}{4}\ln\left|\frac{5-\sin x}{1-\sin x}\right|$ d'où $I = \frac{1}{4}\ln\frac{5-\sin x}{1-\sin x}$

10) $I = \int \frac{dx}{1+3\cos^2 x}$ même remarque que l'exemple 8 et l'exemple 9

$$I = \int \frac{dx}{\cos^2 x\left(\frac{1}{\cos^2 x}+3\right)} = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}+3}\,dx = \int \frac{\frac{1}{\cos^2 x}\,dx}{1+\tan^2 x+3}$$

on pose $t = \tan x \Rightarrow dt = \frac{1}{\cos^2 x}\,dx$ d'où

$$I = \int \frac{dt}{t^2+4} = \int \frac{dt}{4\left(\frac{t^2}{4}+1\right)} = \frac{1}{4}\int \frac{dt}{\left(\frac{t}{2}\right)^2+1} = \frac{1}{2}\cdot\frac{1}{2}\int \frac{dt}{\left(\frac{t}{2}\right)^2+1}$$

on pose $\frac{t}{2} = z \Rightarrow \frac{1}{2}dt = dz$ d'où

$$I = \frac{1}{2}\cdot\int \frac{dz}{z^2+1} = \frac{1}{2}\arctan z = \frac{1}{2}\arctan \frac{t}{2}$$

$$I = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right).$$

11) $I = \int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx = \int \frac{1+1-1-\sin x+\cos x}{1+\sin x-\cos x}\,dx$

$$= \int \frac{2-1-\sin x+\cos x}{1+\sin x-\cos x}\,dx = \int \frac{2-(1+\sin x-\cos x)}{1+\sin x-\cos x}\,dx$$

$$= \int \frac{2}{1+\sin x-\cos x} - \frac{1+\sin x-\cos x}{1+\sin x-\cos x}\,dx$$

$$= \int \frac{2}{1+\sin x-\cos x} - 1\,dx = -x + \int \frac{2}{1+\sin x-\cos x}\,dx$$

↑ Bioche $\Rightarrow$ on pose $t = \tan\frac{x}{2}$

et $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$ ....

.... finalement $I = -x + 2\ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+1}\right|$