République Algérienne Démocratique et Populaire

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

Université Constantine 1

Faculté des Sciences Éxactes

Département de Mathématiques

MATHEMATICS I

Presented by

KESMIA Mounira

FOR STUDENTS IN SCIENCE AND TECHNOLOGY

FIRST SEMESTER 2019-2020

To my students,

from whom I have learned

to explain and how to teach

ACKNOWLEDGMENTS

In the name of Allah, the Most Gracious and the Most Merciful

Alhamdulillah, all praises to Allah for the strengths and His blessing in completing this document.

I thank the mathematics module managers that I worked with them

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Introduction

The Universe is a book whose mathematics would be the language .....

This phrase from Galileo tends to be verified over time and the evolution of mathematical theories. Thus we manage to describe more and more phenomena of the universe whether physical, biological, ecological, economic ... by different laws and mathematical entities.

This handout contains the mathematics I course that I teach in the first semester of the first year of science and technology. I want to point out the principal mathematics tools of algebra and analysis that a student must assimilate and learn. That is, this document can be used as a reference text for undergraduates in the first year in Science and Technology who will be facing mathematics problems and will be interested in learning techniques to solve them.

The course is divided into six chapters cover the basic algebra and analysis, as Propositional Logic, Methods of Proof, Set Theory, Relations, Applications, The inverse Trigonometric Application, Real-valued Functions of Real Variable, Finite Expansions, Vector Spaces and Linear Maps.

I want to inform the reader that I am currently working to improve and expand this text.

Kesmia Mounira

Chapter 1

Preliminaries

Number Sets

In mathematics very often we study sets whose elements are the real numbers. Some special number sets which are frequently encountered are defined as follow.

♦ $\mathbb{N}$ is the set of **Natural numbers**: $\mathbb{N} = \{0, 1, 2, 3, ...\}$
♦ $\mathbb{Z}$ is the set of **Integers:** $\mathbb{Z} = \{....., -3, -2, -1, 0, 1, 2, 3, ...\}$
♦ D is the set of **Decimal numbers:** $D = \left\{\frac{p}{10^n}, p \in \mathbb{Z}, n \in \mathbb{N}\right\}$

Example: $1.234 = \frac{1234}{10^4}$ is a decimal number.

♦ $\mathbb{Q}$ is the set of **Rational numbers:** $\mathbb{Q} = \left\{\frac{p}{q}, p \in \mathbb{Z}, p \in \mathbb{Z}^*\right\}$.

Rational numbers are numbers that can be expressed as the quotient of two integers (ie a fraction) with a denominator that is not zero. Note that all terminating decimals or repeating decimals (or periodic decimal expansion) are a rational numbers.

Examples:

1. $\frac{1}{2} = 0.5$ (terminating decimals).
2. $\frac{9}{7} = 1,285714\mathbf{285714}28571428... = 1,\overline{285714}$ (repeating decimals)

♦ $\Im$ is the set of **Irrational numbers** which are not rational.

Examples: $-\sqrt{3}, \sqrt{2}, \pi$

♦ $\mathbb{R}$ is the set of **Real numbers**, numbers that can be represented by any decimal expansion, limited or not.

Example: 123.101001000100001....,etc.

⧫ $\mathbb{C}$ is the set of **Complex numbers** $\mathbb{Q} = \{a + bi \mid a, b \in R\}$
Recall that a complex number is formed by adding a real number to a real multiple of i, where $i = \sqrt{-1}$.

We have $\mathbb{N} \subset \mathbb{Z} \subset D \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

⧫ The set of **Even numbers** contains the elements $0, \pm 2, \pm 4, \pm 6, ...$ which are those of the form $2n$ for some integer n.

⧫ The set of **Odd numbers** is the set of integers which are not even. Hence odd numbers are $\pm 1, \pm 3, \pm 5, ...$ which can be written as $2n + 1$ for some integer n.

Absolute value

For real numbers x we define the absolute value of x to be

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Examples: $|-2| = 2$, $|\sqrt{2}| = \sqrt{2}$, and $|0| = 0$.

Properties

- $\forall x \in \mathbb{R}, \quad |x| \geq 0.$
- $\forall x \in \mathbb{R}, \quad \sqrt{x^2} = |x|.$
- $\forall x \in \mathbb{R}, |x|^2 = x^2.$
- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |x \cdot y| = |x| \cdot |y|.$
- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad |x + y| \leq |x| + |y|.$

The greatest integer function

For real numbers x, the *greatest integer function* $[x]$ gives the greatest integer not greater than x.

Examples: $[3.14] = 3$, $[-3.14] = -4$, $[0.7] = 0$.

Chapter 2

Logic and mathematical reasoning

2.1 Introduction

Formal logic (symbolic logic) In mathematics, the systematic study of reasoning is called formal logic. It analyzes the structure of ARGUMENTs, as well as the methods and validity of mathematical deduction and proof.

The principles of logic can be attributed to ARISTOTLE (384–322 B.C.E.), who wrote the first systematic treatise on the subject. He sought to identify modes of inference that are valid by virtue of their structure, not their content. For example, "Green and blue are colors; therefore green is a color" and "Cows and pigs are reptiles; therefore cows are reptiles" have the same structure ("A and B, therefore A"), and any argument made via this structure is logically valid. (In particular, the second example is logically sound.) This mode of thought allowed EUCLID (ca. 300–260 B.C.E.) to formalize geometry, using deductive proofs to infer geometric truths from a small collection of AXIOMs (self-evident truths).

No significant advance was made in the study of logic for the millennium that followed. This period was mostly a time of consolidation and transmission of the material from antiquity. The Renaissance, however, brought renewed interest in the topic. Mathematical scholars of the time, Pierre Hérigone and Johann Rahn in particular, developed means for representing logical arguments with abbreviations and symbols, rather than words and sentences. GOTTFRIED WILHELM LEIBNIZ (1646–1716) came to regard logic as "universal mathematics." He advocated the development of a "universal language” or a “universal calculus” to quantify the entire process of mathematical reasoning. He hoped to devise new mechanical symbolism that would reduce errors in thinking to the equivalent of arithmetical errors. (He later abandoned work on this project, assessing it too daunting a task for a single man.)

In the mid-1800s GEORGE BOOLE succeeded in creating a purely symbolic approach to propositional logic, that part which deals with inferences involving simple declarative sentences (statements) joined by the connectives:

$$not, and, or, if \ldots\ then \ldots, iff$$

(These are called the NEGATION, CONJUNCTION, DISJUNCTION, CONDITIONAL, and the BICONDITIONAL, respectively.) He successfully applied it to mathematics, thereby making a significant step to achieving Leibniz’s goal.

In 1879 the German mathematician and philosopher Gottlob Frege constructed a symbolic system for predicate logic. This generalizes propositional logic by including QUANTIFIERs: statements using words such as some, all, and, no. (For example, “All men are mortal” as opposed to “This man is mortal.”) At the turn of the century DAVID HILBERT sought to devise a complete, consistent formulation of all of mathematics using a small collection of symbols with well-defined meanings. English mathematician and philosopher BERTRAND RUSSELL, in collaboration with his colleague ALFRED NORTH WHITEHEAD, took up Hilbert’s challenge. In 1925 they published a monumental work. Beginning with an impressively minimal set of premises (“self-evident” logical principles), they attempted to establish the logical foundations of all of mathematics. This was an impressive accomplishment. (After hundreds of pages of symbolic manipulations, they established the validity of “1 + 1 = 2,” for example.) Although they did not completely reach their goal, Russell and Whitehead’s work has been important for the development of logic and mathematics.

Six years after the publication of their efforts, however KURT GÖDEL stunned the mathematical community by proving Hilbert’s (and Leibniz’s) goal to be futile. He demonstrated once and for all that any formal system of logic sufficiently sophisticated to incorporate basic principles of arithmetic cannot attain all the statements it hopes to prove. His results are today called GÖDEL’S INCOMPLETENESS THEOREMS. The vision to reduce all truths of reason to incontestable arithmetic was thereby shattered.

Understanding the philosophical foundations of mathematics is still an area of intense scholarly research.

2.2 Propositional Logic

Definition (Proposition)

A proposition is a statement which has a truth value either true or false.

Notation: Variables are used to represent propositions. The most common variables used are p, q, and r.

Examples:

1. p :"2 is even", q :"2 + 2 = 4", r :"2 + 2 = 5" are popositions.
2. "$x + 2 = 2x$" is not a proposition.

Definition (Negation)

The negation of a proposition p is also called **not** p, and is denoted by $\bar{p}$.

Examples: Give the negation of the following statements.

1. If p: "2 is even" then $\bar{p}$: "2 is not even".
2. If p: "2 + 2 = 5" then $\bar{p}$: "2 + 2$\neq$ 5".

Definition (Truth-value)

The **truth-value** is one of the two values, "true" (T) or "false" (F), that can be taken by a given logical formula in an interpretation (model) considered. Sometimes the truth value T is denoted in the literature by 1, and F by 0.

2.3 Logical Connectors

2.3.1 Conjunction

Definition (Conjunction)

If p and q are two propositions then their **conjunction** is the proposition whose value is true only when both are true. A conjunction can also be written $p \wedge q$ which is read p **and** q.

Examples:

1. "A triangle has three sides and a square has four sides" is a conjunction
2. Let p : "$2 \leq 3$" and q : "$2^2 \leq 3^2$" , the proposition $p \wedge q$ is true.

2.3.2 Disjunction

Definition (Disjunction)
A compound statement of the form "p **or** q" is known as a disjunction and it is denoted by $p \vee q$. The **disjunction** of p and q has value false only when both are false.
Examples:

1. "An integer is a number which presents itself as a natural integer to which a positive **or** negative sign has been added indicating its position relative to 0 on an oriented axis ".
2. $\bar{p} \vee \bar{q}$: "$2 > 3$" $\vee$ "$2^2 > 3^2$" is a false proposition.

2.3.3 Implication

Definition (Implication)
A **conditional** statement of the form "**If** ... **then**..." is known as a **conditional** or an **implication**.

A conditional statement has two components: If p, then q. Statement p is called the antecedent (hypothesis, or premise) and statement q the consequent (or conclusion).

Alternative Phrasings of Conditionals

A conditional statement can be written a number of different, but equivalent, ways:

If p, then q.
p implies q.
q if p.
p only if q.
p is sufficient for q.
q is necessary for p.
It is denoted in symbols by: $p \Rightarrow q$.

The implication of p and q has value false only when p is true and q is false.
Examples:

1. "If a polygon has three sides, then it is a triangle" is a conditional statement.

2. "If $1 \leq 3$ then $1 + 1 \leq 3 + 1$" is a true implication.

3. "If π and $2 + 3i$ are real numbers then $2 + 3i$ is real number" is a true implication.

4. "If $2 + 3 = 5$ then $3 \times 2 + 3 \times 3 = 20$" is a false implication because when $x = 5$, $3x = 15$ and $15 \neq 20$.

5. "If $(-2)^2 = 4$ then $-2 = \sqrt{4}$" is a false implication because $\sqrt{(-2)^2} \neq -2$.

Definition (Converse of implication)
The converse of $p \Rightarrow q$ is the proposition $q \Rightarrow p$.
Example:
Let p : "x is a prime number different from 2" and q : "x is odd". One has $p \Rightarrow q$ but we do not have $q \Rightarrow p$.

Theorem 1.1: For all propositions p and q, the following statements are true.
1. $p \Rightarrow p \vee q$ and $q \Rightarrow p \vee q$
2. $p \wedge q \Rightarrow p$ and $p \wedge q \Rightarrow q$
Proof
1. We give a truth table for $p \Rightarrow p \vee q$ as follows.

p	q	$p \vee q$	$p \Rightarrow p \vee q$
1	1	1	**1**
1	0	1	**1**
0	1	1	**1**
0	0	0	**1**

Then $p \Rightarrow p \vee q$ is always true.
The truth table for $q \Rightarrow p \vee q$ is analogous to the one for $p \Rightarrow p \vee q$; the conclusion is the same.
2. In order to prove that $p \wedge q \Rightarrow p$ for all propositions p and q, we give a truth table for $p \wedge q \Rightarrow p$

p	q	$p \wedge q$	$p \wedge q \Rightarrow p$
1	1	1	**1**
1	0	1	**1**
0	1	1	**1**
0	0	0	**1**

Then $p \wedge q \Rightarrow p$ is always true.

The truth table for $p \wedge q \Rightarrow q$ is analogous to the one for $p \wedge q \Rightarrow p$.

2.3.4 Equivalence

Definition (Equivalence)

Two mathematical statements are **equivalent** if they have the same truth values.

The statement of the form "p if, and only if, q" is called an **equivalence** or **biconditional** statement. It is often abbreviated as p **iff** q and is written in symbols as $p \equiv q$ or $p \iff q$. It is equivalent to the compound statement "p implies q, and q implies p" composed of two CONDITIONAL statements. The truth-values of p and q must match for the biconditional statement as a whole to be true.

Examples:

1. "A triangle is equilateral if, and only if, it is equiangular" is a biconditional statement.
2. The proposition "$(1 = 1) \iff (0 = 0)$" is true, the proposition "$(1 = 0) \iff (2 = 0)$" is true, whereas the proposition "$(1 = 0) \iff (0 = 0)$" is false.
3. For all real x $(x \neq 0)$ and y , we have $y = x \iff \frac{y}{x} = 1$ is true.
4. The equivalence statement $(x = y \iff x^2 = y^2)$ is not true for all real x and y: for example $2^2 = (-2)^2 \nRightarrow 2 = -2$.

2.4 Truth table

A **truth table** is a table showing the truth-value of a statement (typically a compound one) given the possible truth-values of the simple statements of which it is composed.

The truth values of a proposition, p, can be displayed in tabular form as follows:

p
1
0

The truth-values of the basic connectives are given as follows:

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Longleftrightarrow q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Exercise: Prove the following equivalence by drawing the truth table:

$$p \Rightarrow q \Longleftrightarrow \overline{p} \vee q$$

Solution:

p	q	$\overline{p}$	$p \Rightarrow q$	$\overline{p} \vee q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

The truth table establishes that these corresponding pairs of compound statements are logically equivalent.

Definition (Contrapositive)

The contrapositive of $p \Rightarrow q$ is the proposition $\bar{q} \Rightarrow \bar{p}$. It can be shown that these two are equivalent:

$$(p \Rightarrow q) \Longleftrightarrow (\bar{q} \Rightarrow \bar{p})$$

The equivalence can easily be verified using truth table:

p	q	$\bar{p}$	$\bar{q}$	$p \Rightarrow q$	$\bar{q} \Longleftrightarrow \bar{p}$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

Logical Identities

★ **De Morgan's laws.** $\overline{p \wedge q} \iff \bar{p} \vee \bar{q}$ and $\overline{p \vee q} \iff \bar{p} \wedge \bar{q}$

Both of these laws can easily be verified using truth tables:

p	q	$\overline{p \wedge q}$	$\bar{p} \vee \bar{q}$
1	1	0	0
1	0	1	1
0	1	1	1
0	0	1	1

p	q	$\overline{p \vee q}$	$\bar{p} \wedge \bar{q}$
1	1	0	0
1	0	0	0
0	1	0	0
0	0	1	1

★ **Idempotence of $\wedge$ and $\vee$**

$P \iff P \wedge P$ and $P \iff P \vee P$

★ **Commutativity of $\wedge$ and $\vee$**

$p \wedge q \iff q \wedge p$

$p \vee q \iff q \vee p$

★ **Associativity of $\wedge$ and $\vee$**

$p \wedge (q \wedge r) \iff (p \wedge q) \wedge r$

$p \vee (q \vee r) \iff (p \vee q) \vee r$

★ **Distributivity of $\wedge$ over $\vee$ (and $\vee$ over $\wedge$ respectively)**

$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$

★ **Domination laws**

$P \vee T \iff T$

$P \wedge F \iff F$

★ **Identity laws**

$P \vee F \iff P$

$P \wedge T \iff P$

★ **Negation laws**

$P \vee \bar{P} \iff T$

$P \wedge \bar{P} \iff F$

★ **Double negation law**

$\bar{\bar{p}} \iff p$

★ **Absorption laws**

$P \vee (P \wedge Q) \iff P$

$P \wedge (P \vee Q) \iff P$

Exercise: Prove by applying the above rules.
a) $\overline{p \Rightarrow q} \Longleftrightarrow p \wedge \bar{q}$
b) $\bar{p} \Rightarrow \bar{q} \Longleftrightarrow p \vee \bar{q}$
Solution:
a) By applying De Morgan's laws:

$$\begin{aligned} \overline{p \Rightarrow q} &\Longleftrightarrow \overline{\overline{p} \vee q} \\ &\Longleftrightarrow \overline{\overline{p}} \wedge \bar{q} \\ &\Longleftrightarrow p \wedge \bar{q} \end{aligned}$$

b) By applying the following equivalence statement $p \Rightarrow q \Longleftrightarrow \overline{p} \vee q$, one has:

$$\begin{aligned} \bar{p} \Rightarrow \bar{q} &\Longleftrightarrow \overline{\overline{p}} \vee \bar{q} \\ &\Longleftrightarrow p \vee \bar{q} \end{aligned}$$

Exercise: True or False. Prove by any method you like.
a) $p \Rightarrow (q \Rightarrow r) \Longleftrightarrow (p \Rightarrow q) \Rightarrow r$
b) $p \Rightarrow (q \vee r) \Longleftrightarrow (p \Rightarrow q) \vee (p \Rightarrow r)$
c) $p \wedge (q \Rightarrow r) \Longleftrightarrow (p \wedge q) \Rightarrow (p \wedge r)$
d) $p \vee (q \Rightarrow r) \Longleftrightarrow (p \vee q) \Rightarrow (p \vee r)$
Solution: For example, demonstrate the equivalence statement of (d) using a truth table (you will demonstrate the rest in a similar way)

p	q	r	$q \Rightarrow r$	$p \vee (q \Rightarrow r)$	$p \vee q$	$p \vee r$	$(p \vee q) \Rightarrow (p \vee r)$
1	1	1	1	**1**	1	1	**1**
1	1	0	0	**1**	1	1	**1**
1	0	1	1	**1**	1	1	**1**
1	0	0	0	**1**	1	1	**1**
0	1	1	1	**1**	1	1	**1**
0	1	0	0	**0**	1	0	**0**
0	0	1	1	**1**	0	1	**1**
0	0	0	1	**1**	0	0	**1**

We actually read the same truth values in the fifth and eighth columns.
You'll notice how we filled in the first three columns. This filling method makes it possible to forget no situation.

Exercise: If p and q are true and r and s are false statements, find the truth value of the following statements:

1. $(p \wedge q) \vee r$
2. $p \wedge (r \Rightarrow s)$
3. $(p \vee s) \Leftrightarrow (q \wedge r)$

4. $\overline{p \wedge \overline{r}} \vee (\overline{q} \vee s)$

Solution: Given that p and q are 1 and r and s are 0.

1. $$(p \wedge q) \vee r \Leftrightarrow (1 \wedge 1) \vee 0 \Leftrightarrow 1 \vee 0 \Leftrightarrow 1$$

♦ truth value of the given statement is true.

2. $$p \wedge (r \Rightarrow s) \Leftrightarrow 1 \wedge (0 \Rightarrow 0) \Leftrightarrow 1 \wedge 1 \Leftrightarrow 1$$

♦ truth value of the given statement is true.

3. $$(p \vee s) \Leftrightarrow (q \wedge r) \Leftrightarrow (1 \vee 0) \Leftrightarrow (1 \wedge 0) \Leftrightarrow 1 \Leftrightarrow 0 \Leftrightarrow 0$$

♦ truth value of the given statement is false.

4. $$\overline{p \wedge \overline{r}} \vee (\overline{q} \vee s) \Leftrightarrow \overline{1 \wedge \overline{0}} \vee (\overline{1} \vee 0) \Leftrightarrow 0 \vee 0 \Leftrightarrow 0$$

♦ truth value of the given statement is false.

2.5 Predicates and Quantifiers

Definition (Predicate)

A **predicate** is a statement that contains variables and that may be true or false depending on the values of these variables.

Examples:

1. Let $P(x) : x^2 < x$ is a predicate. One has $P(1) : 1 < 1$ is false and $P(2) : 4 < 2$ is even false. But for $x = \frac{1}{2}$, $P(\frac{1}{2}) : \frac{1}{4} < \frac{1}{2}$ is true.
2. Let $P(x, y) : x^2 + y^2 = (x + y)^2$. Find the values of the following propositions: $P(0, 1)$, $P(0, 0)$, $P(1, 1)$. For which (x, y) is the value of $P(x, y)$ true?

A predicate can also be made a proposition by adding a **quantifier**. There are two quantifiers:

Definition (Universal quantifier)

A **universal quantifier** is a quantifier meaning "for all", "for any", "for each" or "for every", denoted by $\forall$.

Here is a formal way to say that for all values that a predicate variable x can take in a domain A, the predicate is true:

$$\underbrace{\forall\, x}_{\text{for all}} \quad \underbrace{\in A}_{x \text{ belonging to } A} \quad , \quad \underbrace{P(x)}_{P(x) \text{ is true}}$$

Example: All natural numbers of the form $2n+1$ are odd is written: $\forall\, n, 2n+1$ is odd.

Definition (Existential quantifier)

An **existential quantifier** is a quantifier meaning "there exists", "there is at least one" or "for some".

Here is a formal way to say that for some values that a predicate variable x can take in a domain A, the predicate is true:

$$\underbrace{\exists\, x}_{\text{for some}} \quad \underbrace{\in A}_{x \text{ belonging to } A} \quad , \underbrace{P(x)}_{P(x) \text{ is true}}$$

Example: There exists a natural number n satisfying $n \times n = n + n$ can be written: $\exists\, n : n \times n = n + n$.

Remark: A **unique existential quantifier** is a quantifier meaning "there is a unique", "there is exactly one" or "there exists only one". Here is a formal way to say that for some values that a predicate variable x can take in a domain A, the predicate is true:

$$\underbrace{\exists!\, x}_{\text{there exists only one}} \quad \underbrace{\in A}_{x \text{ belonging to } A} \quad , \underbrace{P(x)}_{P(x) \text{ is true}}$$

Example: Let $P(x) : x + 2 = 5$.

1) $\forall x,\ P(x)$: "for all real numbers x, $x+2=5$", which is false.

2) $\exists x,\ P(x)$: "there is a real number x such that $x+2=5$", which is true.

3) $\exists! x,\ P(x)$: "there is a unique real number x such that $x+2=5$", which is true.

Predicate Logic and Negating Quantifiers

We observe, at least intuitively, that the negations of $\exists$ and $\forall$ are correlated in the following manner.

$$\overline{\forall\, x, P(x)} \iff \exists\, x, \overline{P(x)}$$
$$\overline{\exists\, x, P(x)} \iff \forall\, x, \overline{P(x)}$$

Example: There is no natural number n satisfying $n \times n \times n = n+n+n$ as: $\overline{\exists\ n : n \times\ n \times\ n = n + n + n}$

Example: Let $P(x) : x + 2 = 5$.

$\overline{\exists x \in \mathbb{Z}, \text{“}x + 2 = 5\text{”}} \iff \text{“}\forall x \in \mathbb{Z}, x + 2 \neq 5\text{”}$

Exercise: Write the negations by interchanging $\exists$ and $\forall$.

a) There is a real number x such that $x^2 < 0$.

b) Every integer is even.

c) There is an integer x such that $x^2 + 2x + 3 = 0$.

Solution:

a) $\overline{\exists x \in \mathbb{R}, \text{“}x^2 < 0\text{”}} \iff \text{“}\forall x \in \mathbb{R}, x^2 \geq 0\text{”}$

b) There is an integer which is not even.

c) $\overline{\exists x \in \mathbb{Z}, \text{“}x^2 + 2x + 3 = 0\text{”}} \iff \text{“}\forall x \in \mathbb{Z}, x^2 + 2x + 3 \neq 0\text{”}$

Exercise: Write the following propositions with quantifiers :

f is not increasing on $\mathbb{R}$ (where f is a function of $\mathbb{R}$ in $\mathbb{R}$).

Solution: By applying the negation of an implication studied above $[\overline{p \Rightarrow q} \iff p \wedge \bar{q}]$, one has:

$\overline{\forall\ (a, b) \in \mathbb{R}^2 / (a \leq b \Rightarrow f(a) \leq f(b))} \iff \exists\ (a, b) \in \mathbb{R}^2, (a \leq b) \wedge (f(a) > f(b))$

Exercise: Show that the function sin is not zero.

Solution: $\exists x = \frac{\pi}{2}, \sin(\frac{\pi}{2}) = 1 \neq 0$.Then $\sin \neq 0$.

2.5.1 Nested Quantifiers

Two quantifiers are nested if one is within the scope of the other. The order of **existential quantifiers** and **universal quantifiers** in a statement is important.

■ When we have one quantifier inside another, we need to be a little careful.

Example: Consider the following proposition over the integers:

$$\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z}\ /\ (x + y = 0)$$

- The proposition is true.
- The existence of y depends on x : if you pick any x, I can find a y that makes $x + y = 0$ true.

Example: Consider the following proposition over the integers:

$$\exists y \in \mathbb{Z},\ \forall x \in \mathbb{Z}/\ (x + y = 0)$$

- The proposition is false.
- The existence of y does not depend on x : there is no y that will make $x + y = 0$ true for every x.

Example: Consider the following proposition over the integers:

$$\exists y \in \mathbb{Z},\ \forall x \in \mathbb{Z}/\ (x + y = x)$$

- The proposition is true.
- There is $y = 0$ that will make $x + y = x$ for every x.

Example: Suppose we claimed, "For every real number, there's a real number larger than it."

We'd write this as

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y > x$$

- The proposition is true.

■ We can exchange the same kind of quantifier $(\forall, \exists)$.

These statements are equivalent:

$$\begin{aligned} \forall x, \forall y, P(x, y) &\iff \forall y, \forall x, P(x, y) \\ \exists x, \exists y, P(x, y) &\iff \exists y, \exists x, P(x, y) \end{aligned}$$

Exercise: Translate the following statement into a logical expression. "Every real number except zero has a multiplicative inverse."

Solution:

$$\forall y \in \mathbb{R}^*, \exists x \in \mathbb{R} : xy = 1$$

Exercise: Express that the limit of a real-valued function f at point x_0 is l and express its negation: $\lim\limits_{x \to x_0} f(x) = l$.

Solution:

In predicate logic:

$$\forall \varepsilon > 0, \exists\ \delta > 0, \forall x \in \mathbb{R} : (|x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon)$$

its negation is given by: $\exists\ \varepsilon > 0, \forall \delta > 0, \exists\ x \in \mathbb{R} : |x - x_0| < \delta \wedge |f(x) - l| \geq 0$.

Exercise: If $A = \{3, 4, 6, 8\}$, determine the truth value of each of the following:

1. $\exists\ x \in A$, $x + 4 = 7$.
2. $\exists\ x \in A$, x is odd.
3. $\forall\ x \in A$, $(3 - x) \in \mathbb{N}$.

Solution:

1. Since $x = 3 \in A$, satisfies $x + 4 = 7$, the given statement is true. Its truth value is '1'.
2. Since $x = 3 \in A$, satisfies the given statement, the given statement is true. Its truth value is '1'.
3. $\exists\ x \in A, x = 4$, do not satisfy $3 - 1 = -1 \notin \mathbb{N}$,the given statement is false. Its truth value is '0'.

2.6 Methods of Proof

Our main interest in quantifiers for the purposes of this course is to develop techniques for proving mathematical statements.

When faced with a mathematical claim, understanding its quantifier is often a very good strategy for thinking about how to work out a proof.

Example: If the statement has the form $\forall x : P(x)$, then the global outline is likely have the form: Consider any possible x, and show that it satisfies the property $P(x)$.

Example: If the statement has the form $\exists x : P(x)$, then the global outline is different: One needs to specify a particular x, and then show it satisfies $P(x)$.

2.6.1 Direct Methods

We have already seen one way of proving a mathematical statement of the form: If p, then q. Based of the fact that the implication $p \Longrightarrow q$ is false only when p is true and q is false, the idea behind the method of proof that we discussed was to assume that p is true and then to proceed, through a chain of logical deductions, to conclude that q is true. Here is the outline of the argument:

Suppose that p is true.

$$\begin{aligned} p &\Rightarrow r \\ &\Rightarrow s \\ &\Rightarrow .. \\ &\Rightarrow q \end{aligned}$$

Exercise: Prove the statement: If n is even, then n^2 is even.

Solution: Assume that the integer n is even.

$$\begin{aligned} \exists\ k \in \mathbb{Z}, n = 2k &\Longrightarrow & n^2 = (2k)^2 = 4k^2 \\ &\Longrightarrow & n^2 = 2(2k^2) \\ &\Longrightarrow & n^2 = 2k^{'} \text{ such that } k^{'} = 2k^2 \end{aligned}$$

which shows that n^2 is even.

This is an example of a **direct method** of proof. In the following section we discuss **indirect methods** of proof.

2.6.2 Proof by Contrapositive

The idea behind this method of proof comes from the fact that the implication

$$\bar{q} \Rightarrow \bar{p}$$

is equivalent to the implication

$$p \Rightarrow q$$

Thus, in order to prove $p \Rightarrow q$, it suffices to prove: $\bar{q} \Rightarrow \bar{p}$. Here is the outline of the argument:

Suppose that : $\bar{q}$ is true

$$\begin{aligned} \bar{q} &\Longrightarrow r \\ &\Longrightarrow s \\ &\Longrightarrow ... \\ &\Longrightarrow \bar{p}. \end{aligned}$$

Consequently, $\bar{q} \Rightarrow \bar{p}$ is true; therefore, $p \Rightarrow q$ is true.

Exercise: Prove that n^2 is even implies that n is even.
Solution: Suppose that n is not even. It then follows that

$$\begin{aligned}
\exists\, k \in \mathbb{Z}, n = 2k+1 \quad &\Longrightarrow \quad n^2 = (2k+1)^2 = 4k^2 + 4k + 1 \\
&\Longrightarrow \quad n^2 = 2(2k^2 + 2k) + 1 \\
&\Longrightarrow \quad n^2 = 2k^{'} + 1 \text{ such that } k^{'} = 2k^2 + 2k
\end{aligned}$$

Thus, n is not even implies that n^2 is not even, and therefore the contrapositive is true; namely, n^2 is even implies that n is even.

2.6.3 Proof by Contradiction (Absurd)

To prove that a proposition p is true we may assume that p is false then $\bar{p}$ is true. Therefore we show that it would lead to a **contradiction** or a false statement.

Exercise: Prove that $\sqrt{2}$ is irrational.

Solution: Let p: $\sqrt{2}$ is irrational. Now assume that p is false then $\bar{p}$ is true, that is, $\sqrt{2}$ is rational. Then there are some integers a and b with no common factors:

$$\begin{aligned}
\exists\, a \in \mathbb{Z}, \exists\, b \in \mathbb{Z}^* : \sqrt{2} = a/b \quad &\Rightarrow \quad a^2 = 2b^2 \\
&\Rightarrow \quad a^2 \text{ is even} \\
&\Rightarrow \quad a = 2c, \quad c \in \mathbb{Z} \\
&\Rightarrow \quad 4c^2 = 2b^2 \text{ (by Substituting)} \\
&\Rightarrow \quad 2c^2 = b^2 \\
&\Rightarrow \quad b^2 \text{ is even} \\
&\Rightarrow \quad b \text{ is even}
\end{aligned}$$

This means that a and b have a common factor 2 which is a contradiction, and so $\bar{p}$ must be false and p is true.

2.6.4 Proof by Counter-Example

This proof structure allows us to prove that a property is not true by providing an example where it does not hold. Thus, in order to prove that the statement $\forall\, x$, $P(x)$ is false, it suffices to prove that the statement $\exists\, x, \overline{P(x)}$ is true.

Exercise: Prove that "all triangles are obtuse" is false.

Solution: We give the following counter example: the equilateral triangle having all angles equal to sixty. In this case, there are infinitely many counter example. However, it only takes one.

Exercise: Prove that " If n is an integer and n^2 is divisible by 4, then n is divisible by 4" is false.

Solution: Consider $n = 6$. Then $n^2 = 36$ is divisible by 4, but $n = 6$ is not divisible by 4. Thus, $n = 6$ is a counter example to the statement.

Exercise: Prove that "$(a + b)^2 = a^2 + b^2$" is not an algebraic identity, where $a, b \in \mathbb{R}$.

Solution: If $a = 1$ and $b = 2$, then $(a+b)^2 = 9$ and $a^2+b^2 = 1^2+2^2 = 5$.

2.6.5 Proof by Cases Disjunction

This proof structure is used when one wants to prove a property $\forall\ x,\ P(x)$ depending on a parameter x belonging to a set E, and the proof depends on the value of x. Hence we decompose the set E into two or more sets $E_1, E_2, ...$ and we separate the reasonings following that $x \in E_1, x \in E_2, ...$ This proof is often used to solve (in) equations with absolute values (the proof depends on the sign of the quantity within the absolute value), to demonstrate properties in arithmetic (we separate the proof following the parity of some integers, their congruence modulo n ...).

To prove a proposition by case in the form $p \Rightarrow q$ where $p \iff r \vee s$ we may instead prove both $r \Rightarrow q$ and $s \Rightarrow q$.

Exercise: Prove that for any integer n,the quotient $\frac{n(n+1)}{2}$ is an integer.
Solution:

- If n is even, then n is written $n = 2k$ and $n + 1 = (2k + 1)$. We then have $\frac{n(n+1)}{2} = k(2k + 1)$ which is an integer.
- If n is odd, then n is written $n = 2k + 1$ and $n + 1 = 2k + 2$. We then have $\frac{n(n+1)}{2} = (2k + 1)(k + 1)$ which is also an integer.

Exercise: Prove that $\forall x \in \mathbb{R} : |x - 1| \leq x^2 - x + 1$.

Solution: $|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x+1 & \text{if } x < 1 \end{cases}$

$$|x-1| \leq x^2 - x + 1 \Leftrightarrow \begin{cases} x-1 \leq x^2 - x + 1 & \text{if } x \geq 1 \\ -x+1 \leq x^2 - x + 1 & \text{if } x < 1 \end{cases}$$

1. If $x \geq 1$, $x - 1 - x^2 + x - 1 \leq 0 \Leftrightarrow -x^2 + 2x - 2 \leq 0$ is true because the discriminant of the equation $x^2 - 2x + 2 = 0$ is negative ($\triangle = -4$), hence $x^2 - 2x + 2 \geq 0$

2. If $x < 1, -x + 1 - x^2 + x - 1 \leq 0 \Leftrightarrow -x^2 \leq 0$ is true.

Therefore $\forall x \in \mathbb{R} : |x - 1| \leq x^2 - x + 1$.

2.6.6 Proof by Mathematical Induction

To prove a proposition in the form $\forall n \in \mathbb{N}, P(n)$ where n is a natural number, it suffices to prove it in two steps:

1. $P(n_0)$ is true for a certain base step n_0. Usually the base case is $n = 1$ or $n = 0$.
2. $P(n) \Rightarrow P(n + 1)$. That is, if $P(n)$ is true, then $P(n + 1)$ is true.

Exercise: Prove the following formula for all natural numbers n.
$1 + 3 + 5 + 7 + 9 + ... + (2n - 1) = n^2$.

Solution: Let $P(n) : 1 + 3 + 5 + 7 + 9 + ... + (2n - 1) = n^2$
We shall prove $\forall n \in \mathbb{N}, P(n)$ in two steps:
1) $P(0) : 0 = 0^2$ so this proposition is true.
2) Let $P(n) : 1 + 3 + 5 + 7 + 9 + ... + (2n - 1) = n^2$

$$\Rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2n - 1) + (2n + 1) = n^2 + (2n + 1)$$
$$\Rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2n - 1) + (2n + 1) = (n + 1)^2$$
$$\Rightarrow 1 + 3 + 5 + 7 + 9 + ... + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$$
$$\Rightarrow P(n + 1) \text{ is true}$$

Therefore $\forall n \in \mathbb{N}, P(n)$
Exercise: is $3^n - 1$ a multiple of 2 ?
Solution:

1. Show it is true for $n = 1$, $3^1 - 1 = 3 - 1 = 2$. One has 2 is a multiple of 2. That was easy. $3^1 - 1$ is true
2. Assume it is true for n and prove that $3^{n+1} - 1$ is a multiple of 2?;

$$\begin{aligned} 3^n - 1 = 2k \quad &\Rightarrow \quad 3^n \times 3 - 1 \times 3 = 2k \times 3 \\ &\Rightarrow \quad 3^n \times 3 - 3 = 2k \times 3 \\ &\Rightarrow \quad 3^{n+1} - 1 = 2 + 2k \times 3 \\ &\Rightarrow \quad 3^{n+1} - 1 = 2(1 + 3k) \\ &\Rightarrow \quad 3^{n+1} - 1 = 2k' \text{ such that } k' = 1 + 3k \end{aligned}$$

Therefore $\forall n \in \mathbb{N},\ 3n - 1$ a multiple of 2

Chapter 3

Sets, Relations and Applications

3.1 Set Theory

Set is a very basic concept used in all branches of mathematics and computer science. Although the intuitive notion of a set as a collection of objects is as ancient as the human race, the idea of a set as a formal mathematical concept was not proposed until the 19th century. In his development of BOOLEAN ALGEBRA, British mathematician GEORGE BOOLE (1815–64) introduced the notion of set as a fundamental tool for the study of FORMAL LOGIC. German mathematician GEORG CANTOR (1845–1918), in his attempts to understand the foundation of all of mathematics, came to regard sets as even more basic and fundamental than the notion of "number." Cantor properly formalized a theory of set manipulations and introduced the striking notion of CARDINALITY. His work led him to profound insights into the nature of finite and infinite sets, leading him to extend the concept of number to include more than one type of INFINITY.

3.1.1 Relationships between elements and parts of a set

Definition (Set, Element)
A **set** A is a collection of objects called the **elements** of the set.

- If x is an element of the set A then we write $x \in A$, while the negation is written $x \notin A$.

- Set is typically specified either *explicitly*, that is by listing all the elements the set contains, or *implicitly*, using a predicate description as seen in predicate logic, of the form $\{x| \ P(x)\}$.
- The ordering of the elements is not important and repetition of elements is ignored.
- A set may also be **empty** set (or **null set**) and it is denoted by $\emptyset$ (phi) or $\{\}$.
- The **universe** E is the biggest set in which all the other sets we are interested in lie.

Examples:

1. The set A given by $\{1, 2, 3\}$ is an explicit description.
2. The set $\{x, \ \ x$ is a prime number $\}$ is implicit.
3. $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ is a set containing five sets.
4. $\{x : x \in \{2, 3, 5\}$ and $x \leq 1\}$ is an empty set.
5. $\{x : x^2 = -1 \ \}$ is the set of two elements: i and $-i$.
6. $\{e, \pi, 1, \pi, 2, 1\} = \{1, 2, \pi, e\}$

Definition (Cardinality)

If a set A contains exactly n elements where n is a non-negative integer, then A is a finite set, and n is called the **cardinality** of A. We write $|A| = n$.

Remark: If $|A|$ is finite, A is a finite set; otherwise, A is infinite.

Examples:

1. $A = \{1, 2, \sqrt{7}, 0\}, |A| = 4$.
2. $|\{x \mid \ -2 < x < 5, \, x \in \mathbb{Z}\}| = 6$.
3. $|\emptyset| = 0$.
4. $|\{x \mid (x \in \emptyset) \wedge (x < -4)\}| = 0$.
5. The set of positive integers is an infinite set.

Using set notation with quantifiers

Sometimes, we restrict the domain of a quantified statement explicitly by using set notations.

- We use $\forall\, x \in A$, $(P(x))$ to denote that $P(x)$ **holds for every** $x \in A$.
- We use $\exists\, x \in A$, $(P(x))$ to denote that $P(x)$ **holds for some** $x \in A$.

Definition (Equality)

Two sets A and B are **equal** if each element of A is an element of B and vice versa. This is denoted, $A = B$. Formally:

$$\boxed{A = B \Leftrightarrow \forall x : x \in A \Leftrightarrow x \in B} \tag{2.1}$$

Examples:

1. $\{1, 2, 3\} = \{2, 1, 3\}$.
2. $\{1, 2, 3, 4\} = \{x \in \mathbb{N}\ , x < 5\}$.
3. $\{x \in \mathbb{R} : x^2 + 1 = 0\} = \emptyset$.

To say that two sets A and B are *not equal*, inequality is $A \neq B$ of course. We use the negation from predicate logic, which is (using the rules we have studied in predicate logic! namely negation of universal quantifier and De Morgan's law). One obtains:

$$\begin{aligned}
\overline{\forall x : x \in A \Leftrightarrow x \in B} &\Leftrightarrow \exists\, x : \overline{(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)} \\
&\Leftrightarrow \exists\, x : \overline{(x \in A \Rightarrow x \in B)} \vee \overline{(x \in B \Rightarrow x \in A)} \\
&\Leftrightarrow \exists\, x : \left(\overline{\overline{(x \in A)} \vee (x \in B)}\right) \vee \left(\overline{\overline{(x \in B)} \vee (x \in A)}\right) \\
&\Leftrightarrow \exists\, x : \left((x \in A) \wedge \overline{(x \in B)}\right) \vee \left((x \in B) \wedge \overline{(x \in A)}\right) \\
&\Leftrightarrow \exists\, x : ((x \in A) \wedge (x \notin B)) \vee ((x \in B) \wedge (x \notin A))
\end{aligned}$$

Definition (Subset)

- A set A is a **subset** of B *if and only if* every element of A is also in B. We use $A \subseteq B$ to indicate A is a subset of B , that means A is **included** in B. Formally

$$\boxed{A \subseteq B \Leftrightarrow \forall x : x \in A \Rightarrow x \in B} \tag{2.2}$$

- A is a **proper subset (or strict subset**) of B, $A \subset B$, if $A \subseteq B$ and $A \neq B$.

- Note the difference between $x \in A$ and $\{x\} \subseteq A$: in the first expression, x is an *element* of A, while in the second, we consider the *subset* $\{x\}$, which is emphasized by the bracket notation.

- To say that A is *not a subset* of B, we use the negation of the following statement $[\forall x : x \in A \Rightarrow x \in B]$, which is:

$$\overline{\forall x : x \in A \Rightarrow x \in B} \quad \Leftrightarrow \quad \exists\, x : (x \in A) \wedge (x \notin B)$$

Therfore,

$$\boxed{A \nsubseteq B \Leftrightarrow \exists\, x : (x \in A) \wedge (x \notin B)} \tag{2.3}$$

Examples:

1. $\{12, 43, 66\} \subseteq \{12, 43, 66\}$
2. $\{a, \bigstar\} \subset \{a, b, \bigstar, \blacktriangle\}$
3. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
4. Let $A = \left\{\sqrt{2}, i\right\}$, $A \nsubseteq \mathbb{R}$; $i \in A$ and $i \notin \mathbb{R}$

Remark: There is a difference between $\emptyset$ and $\{\emptyset\}$: the first one is an empty set, the second one is a set, which is not empty since it contains one element: the empty set!

Properties: Notice that $A \subseteq A$ and in fact each set is a subset of itself. The empty set $\emptyset$ is a subset of any set $\emptyset \subseteq A$.

Exercise: Prove that $\emptyset \subseteq A$

Solution: Recall the definition of a subset: all elements of a set A must be also elements of B; $\forall x : x \in A \Rightarrow x \in B$. We must show the following implication is true for any $A, \forall x : x \in \emptyset \Rightarrow x \in A$. Since the empty set does not contain any element, $x \in \emptyset$ is *always False statement*. Then the implication is *always True*.

Venn Diagram

A diagram in which mathematical sets are represented by overlapping circles within a boundary representing the **universal set** is called a **Venn diagram**. Such diagrams provide convenient pictorial representations of relations between sets.

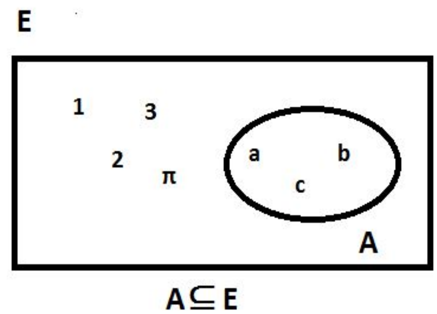


Figure 3.1

Example: In the diagram (See Figure 3.1) a **universal set** E is represented by the interior of a rectangle, and one subset A of E as the interior of one overlapping circle within the rectangle.

The Venn diagram in Fig.1 shows: $A \subseteq E$, $\{1, 2, 3, \pi\} \subseteq E$, $\{1, 2, 3, \pi\} \nsubseteq A$, ...

Definition (Power set)

Given a set E, the power set of E is the set of all subsets of E. The power set is denoted by $P(E)$. Formally:

$$\boxed{P(E) = \{A,\ A \subseteq E\ \}} \tag{2.4}$$

Examples: Write the power set of the following sets: $\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}$. If E is a set with $|E| = n$ then $|P(E)| =$?

1. $P(\emptyset) = \{\emptyset\}$ and $|P(\emptyset)| = 1$.
2. $P(\{1\}) = \{\emptyset, \{1\}\}$ and $|P(\{1\})| = 2$.
3. $P(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $|P(\{1, 2\})| = 4$.
4. $P(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ and $|P(\{1, 2, 3\}| = 8$.

Property: If E is a set with $|E| = n$ then $|P(E)| = 2^n$.

3.1.2 Set Operations

There are a number of basic set manipulations, each of which can be depicted with a VENN DIAGRAM.

Definition (Set Intersection)

The **intersection** of the sets A and B is the set of all elements that are in both A and B. We write $A \cap B$. Formally:

$$\boxed{A \cap B \quad = \quad \{x, (\ x \in A) \ \ \wedge \ \ (x \in B)\}} \tag{2.5}$$

Venn Diagram of Intersection Operation (See figure 3.2):

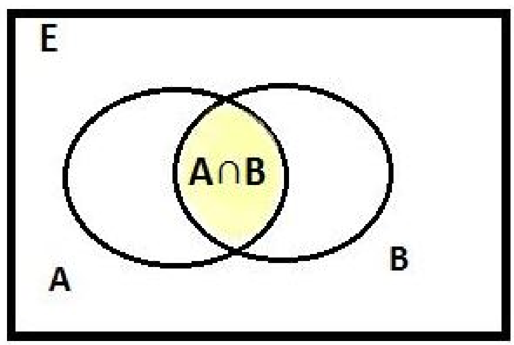


Figure 3.2

Example:

1. $\{1, 2, 3, 4\} \cap \{-3, 4, 5\} = \{4\}$
2. $\{x|\ x > 0\} \cap \{x|\ x \geq 2\} = \{x|\ x \geq 2\}$.
3. $\mathbb{N} \cap \mathbb{Z} \cap \mathbb{R} = \mathbb{N}$

Definition (Set Disjoint)

Two sets A and B are disjoint if $A \cap B = \emptyset$.

Examples:

1. $\{2, 4, 6\} \cap \{8, 10, 12\} = \emptyset$, so they are disjoint.
2. $\{1, 2, 3\} \cap \{3, 4, 5\} \neq \emptyset$, so they are not disjoint.
3. $\mathbb{N} \cap \mathbb{Z} \neq \emptyset$, so they are not disjoint.

4. $\{x|\ x \geq 1\} \cap \mathbb{Z}^- = \emptyset$, so they are disjoint.

Definition (Set Union)

The **union** of two sets A and B , denoted by $A \cup B$, is the set that contains exactly all the elements that are in either A or B (or in both). Formally:

$$\boxed{A \cup B \;=\; \{x,\ (x \in A)\ \vee\ (x \in B)\}} \tag{2.6}$$

Venn Diagram of Union Operation (See Figure 3.3):

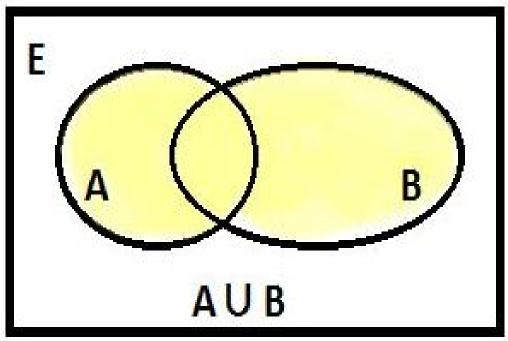


Figure 3.3

Examples:

1. Let $A = \{0, 1, 2, 3, 6\}$, $B = \{0, 1, 2, 4, 6, 9\}$, $A \cup B = \{0, 1, 2, 3, 4, 6, 9\}$.
2. $\mathbb{Z}^- \cup \mathbb{Z}^+ = \mathbb{Z}$.
3. $\{x|\ x > 0\} \cup \{x|\ x > -1\} = \{x|\ x > -1\}$.

Lemma (Cardinality of intersection and union)
For any two sets A and B, we have

$$\boxed{|A \cup B| \;=\; |A| + |B| - |A \cap B|} \tag{2.7}$$

Theorem 2. If A and B are any sets, then

1. $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$

2. $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$

Proof

1) To prove that $A \cap B \subseteq A$, we must show that $x \in A \cap B \Rightarrow x \in A$.

$$\begin{aligned} x \in A \cap B &\Rightarrow (x \in A) \wedge (x \in B) && \text{by (2.5)} \\ &\Rightarrow x \in A && \text{by theorem 1} \end{aligned}$$

Analogously, we can show that $(A \cap B) \subseteq B$.

2) To prove that $A \subseteq A \cup B$, we must show that $x \in A \Rightarrow x \in A \cup B$:

$$\begin{aligned} x \in A &\Rightarrow (x \in A) \vee (x \in B) && \text{by theorem 1} \\ &\Rightarrow x \in A \cup B && \text{by (2.6)} \end{aligned}$$

Analogously, we can show that $B \subseteq A \cup B$

Properties: For all subsets A, B and C of the univers E , the following are true.

- **Commutative laws:** $\begin{cases} A \cap B = B \cap A \\ A \cup B = B \cup A \end{cases}$
- **Associative laws:** $\begin{cases} (A \cap B) \cap C = B \cap (A \cap C) \\ (A \cup B) \cup C = B \cup (A \cup C) . \end{cases}$
- **Distributive laws:** $\begin{cases} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{cases}$
- **Identity laws:** $\begin{cases} A \cup \emptyset = A \\ A \cap E = A \end{cases}$
- **Domination laws:** $\begin{cases} A \cup E = E \\ A \cap \emptyset = \emptyset \end{cases}$
- **Idempotent laws:** $\begin{cases} A \cup A = A \\ A \cap \emptyset = \emptyset \end{cases}$

Definition (Set Partition)

A collection of nonempty sets $\{A_1, A_2, ..., A_n\}$ is a partition of a set A if and only if

1. $A = A_1 \cup\ A_2 \cup ...\ \cup\ A_n$.

2. $A_1,\ A_2, \cdots,\ A_n$ are mutually disjoint (or pairwise disjoint) : $Ai \cap Aj = \emptyset$, $i \neq j$, $i, j = 1, 2, .., n$.

Example: Consider $A = \mathbb{Z}$, $A_1 = \{x,\ x \text{ is even}\}$, $A_2 = \{x,\ x \text{ is odd}\}$. Then A_1, A_2 form a partition of A.
Venn Diagram of Set partition (See Figure 3. 4):

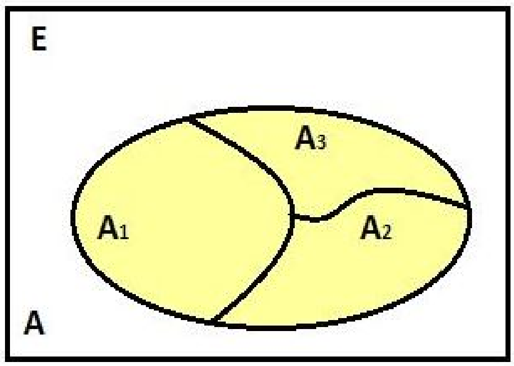


Figure 3.4

Definition (Set difference)

The **difference** of A and B, is the set containing elements that are in A but not in B. Formally:

$$\boxed{A - B \quad = \quad \{x,\ (x \in A) \wedge (x \notin B)\}} \qquad (2.8)$$

Venn Diagram of Set difference (See Figure 3.5):
Examples:

1. $\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$.

2. $\mathbb{R} - \{0\} = \{x |\ (x \in \mathbb{R}) \wedge (x \neq 0)\}$.

3. $\mathbb{N} - \left\{\frac{a}{b}, (a \in \mathbb{Z}) \wedge (b \in \mathbb{Z}^*)\right\} = \mathbb{N}$.

Properties: Let A and B subsets of the univers E.

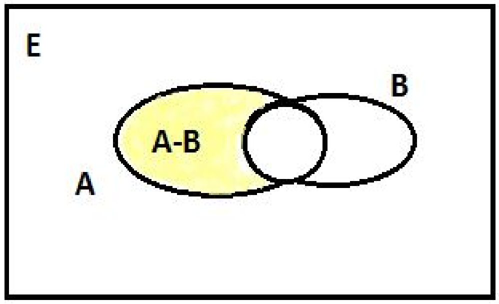


Figure 3.5

- $A - B \subset A$
- $A - A = \emptyset, \quad A - \emptyset = A, \quad \emptyset - A = \emptyset$

Definition (Set complement)
Let A subset of the universal set E. The complement of set A with respect to E, denoted by C_E^A or CA or $\overline{A}$, is the set that contains exactly all the elements that are not in A. Formally:

$$\boxed{\overline{A} = E - A = \{x \in E / \ x \notin A\}} \tag{2.9}$$

Venn Diagram of Set complement (See Figure 3.6):

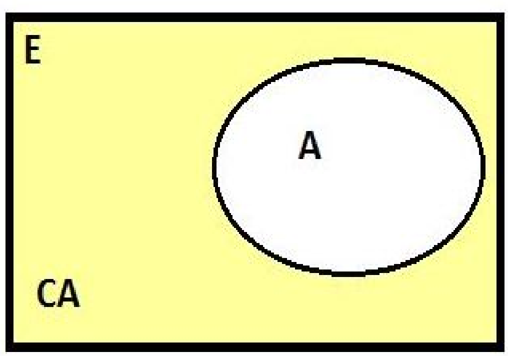


Figure 3.6

Examples: Let the universe be $\mathbb{R}$

1. $\overline{\{0\}} = \{x,\ x \neq 0\} = \mathbb{R}^*$.
2. $\overline{\mathbb{R}^-} = \{x,\ x > 0\} = \mathbb{R}^+$.
3. $\overline{]-1,2]} =]-\infty,-1] \cup]2,+\infty[$.
4. $\overline{\mathbb{R}} = \{x,\ x \notin \mathbb{R}\} = \emptyset$.

Properties: Let A and B subsets of the univers E.

- $\overline{E} = C_E^E = \emptyset, \quad \overline{\emptyset} = C_E^{\emptyset} = E$
- $C_E\left(C_E^A\right) = \overline{\overline{A}} = A$
- $A \cup \overline{A} = \emptyset, \quad A \cap \overline{A} = \emptyset$
- $A \subset B \implies \overline{B} \subset \overline{A}$

DE MORGAN'S LAWS explain how set complement interacts with intersections and unions of sets.

$$\begin{cases} \overline{A \cap B} = \overline{A} \cup \overline{B} \\ \overline{A \cup B} = \overline{A} \cap \overline{B} \end{cases}$$

Exercise: Let A and B are subsets of the universal set E. Show that:

1. $C_E\left(C_E^A\right) = \overline{\overline{A}} = A$
2. $A \subset B \implies \overline{B} \subset \overline{A}$
3. $\overline{A \cap B} = \overline{B} \cup \overline{A}$
4. $\overline{A \cup B} = \overline{B} \cap \overline{A}$

Solution:
1. $C_E\left(C_E^A\right) = \overline{\overline{A}} = A$?
$x \in C_E\left(C_E^A\right) \iff x \notin \left(C_E^A\right) \iff x \in A$
2. $A \subset B \implies \overline{B} \subset \overline{A}$?

$$\begin{aligned} A \subset B &\iff (\forall x, x \in A \implies x \in B) \\ &\iff (\forall x, x \notin B \implies x \notin A) \\ &\left(\forall x, x \in \overline{B} \implies x \in \overline{A}\right) \\ &\overline{B} \subset \overline{A} \end{aligned}$$

3. $\overline{A \cap B} \;=\; \overline{B} \cup \overline{A}$

$$
\begin{aligned}
x \in \overline{A \cap B} &\iff x \notin (A \cap B)\\
&\iff (x \notin A) \vee (x \notin B)\\
&\iff \big(x \in \overline{A}\big) \vee \big(x \in \overline{B}\big)\\
&\iff x \in \overline{A} \cup \overline{B}
\end{aligned}
$$

4. $\overline{A \cup B} \;=\; \overline{B} \cap \overline{A}$

$$
\begin{aligned}
x \in \overline{A \cup B} &\iff x \notin (A \cup B)\\
&\iff (x \notin A) \wedge (x \notin B)\\
&\iff \big(x \in \overline{A}\big) \wedge \big(x \in \overline{B}\big)\\
&\iff x \in \overline{A} \cap \overline{B}
\end{aligned}
$$

Definition (Set Symmetric Difference)

The **symmetric difference** of set A and set B, denoted by $A \triangle B$, is the set containing those elements in exactly one of A and B.

Formally:

$$\boxed{A \triangle B \;=\; (A - B) \cup (B - A)} \tag{2.10}$$

Venn Diagram of Set difference (See Figure 3.7):

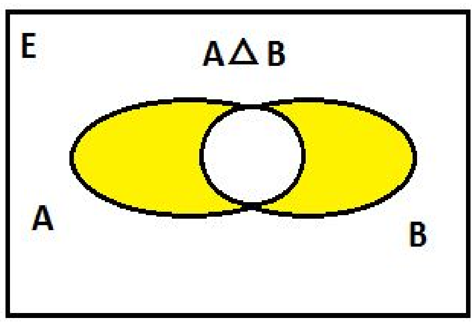


Figure 3.7

Properties: Let A and B subsets of the univers E.

- $A \triangle B \;=\; B \triangle A$
- $A \triangle \emptyset \;=\; A$
- $A \triangle E \;=\; E - A$
- $A \triangle A \;=\; \emptyset$

- $A \triangle B \quad = \quad (A \cup B) - (A \cap B)$
- $A \triangle B \quad = \quad \overline{A} \triangle \overline{B}$

Example: If $A = \{1, 2, 3, 4, 5, 10\}$ and $B = \{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, then $A - B = \{10\}$ and $B - A = \{0, 7, 8, 9\}$. Hence $A \triangle B = \{0, 7, 8, 9, 10\}$

Definition (Ordered tuple)

An **ordered n-tuple** $(x_1, x_2, ..., x_n)$ has x_1 as its first element, x_2 as its second element, . . ., x_n as its nth element. The order of elements is important in such a tuple. Note that $(x_1, x_2) \neq (x_2, x_1)$ but $\{x_1, x_2\} = \{x_2, x_1\}$.

Definition (Set Cartesian product)

The **Cartesian product** of the sets A and B, denoted by $A \times B$ is the set of all ordered pairs (x_1, x_2), where $x_1 \in A$, $x_2 \in B$:

$$\boxed{A \times B = \{(x_1, x_2)/ \quad x_1 \in A, x_2 \in B\ \}} \tag{2.11}$$

The equality in $A \times B$ is defined by: $(x_1, y_1) = (x_2, y_2) \Longleftrightarrow x_1 = x_2 \wedge y_1 = y_2$.

Cartesian product can be formed from n sets $A_1, A_2, ..., A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is defined as the set of ordered tuples $(x_1, x_2, ..., x_n)$ where $x_1 \in A_1, x_2 \in A_2, ..., x_n \in A_n$. That is:

$$\boxed{A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, .;, x_n)/\ x_1 \in A_1, x_2 \in A_2, .., x_n \in A_n\}} \tag{2.12}$$

$$\boxed{A_1 = A_2 = \cdots = A_n = A \quad \Longrightarrow \quad A_1 \times A_2 \times \cdots \times A_n = A^n} \tag{2.12}$$

If we represent a set $A \times B$, then a segment of the horizontal axis is marked off to represent A and a segment of the vertical axis is marked off to represent B; $A \times B$ is the rectangle determined by these two segments (See Figure 8).

Examples: Let $A = \{-2, 3\}$ and $B = \{0, -4, 2\}$

1. $A \times B = \{(-2, 0), (-2, -4), (-2, 2), (3, 0), (3, -4), (3, 2)\}$.
2. $B \times A = \{(0, -2), (0, 3), (-4, -2), (-4, 3), (2, -2), (2, 3)\}$

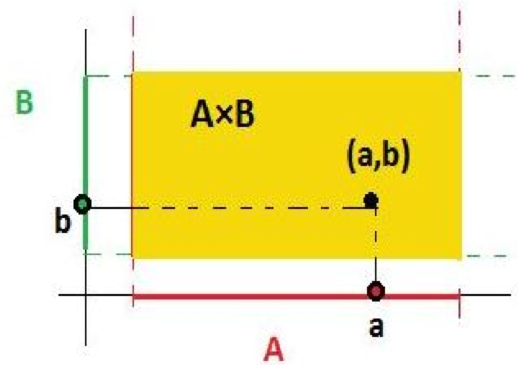


Figure 3.8

3. Note that $A \times B \neq B \times A$.

4. $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$ is the set of point coordinates in the 2D Plane.

5. $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$ is the set of point coordinates in the 3D Space.

Lemma (**Cardinality of Cartesian product**)

In general, if A_i's are finite sets, we have:

$$\boxed{|A_1 \times A_2 \times ... \times A_n| = |A_1| \times |A_2| \times ... \times |A_n|} \tag{2.13}$$

Properties: If A and B are any sets, then

- $A \times B \neq B \times A$
- $A \times \emptyset = \emptyset \times A = \emptyset$

Exercise : The sets $A = \{1, 2, x\}$, $B = \{3, 4, y\}$ are given. Determine x and y, knowing that $\{1, 3\} \times \{2, 4\} \subseteq A \times B$

Solution: We form the sets $A \times B$ and $C = \{1, 3\} \times \{2, 4\}$:
$A \times B = \{(1, 3), (1, 4), (1, y), (2, 3), (2, 4), (2, y), (x, 3), (x, 4), (x, y)\}$

$C = \{(1, 2), (1, 4), (3, 2), (3, 4)\}$

Because $\{1,3\} \times \{2,4\} \subseteq A \times B$, we obtain
$(1,2) \in C \Longrightarrow (1,2) \in A \times B \Longrightarrow (1,2) = (1,y) \Longrightarrow y = 2.$
$(3,4) \in C \Longrightarrow (3,4) \in A \times B \Longrightarrow (3,4) = (x,4) \Longrightarrow x = 3.$

For $x = 3$ and $y = 2$, we have $(3,2) \in A \times B$.
Therefore: $x = 3$ and $y = 2$.

Exercise : Determine the sets A and B that simultaneously satisfy the following the conditions:

1. $A \cup B = \{1,2,3,4,5\}$;
2. $A \cap B = \{3,4,5\}$;
3. $2 \notin (B - A)$
4. $1 \notin (A - B)$

Solution:
$1 \notin (A - B) \Leftrightarrow (1 \notin A) \vee (1 \in B)$
$1 \in (A \cup B) \Leftrightarrow (1 \in A) \vee (1 \in B)$

$$\begin{aligned}
[1 \notin (A - B)] \wedge [1 \in (A \cup B)] &\Leftrightarrow [(1 \notin A) \vee (1 \in B)] \wedge [(1 \in A) \vee (1 \in B)] \\
&\Leftrightarrow [(1 \notin A) \wedge (1 \in A)] \vee (1 \in B) \\
&\Leftrightarrow F \vee (1 \in B) \\
&\Leftrightarrow 1 \in B
\end{aligned}$$

$2 \notin (B - A) \Leftrightarrow (2 \notin B) \vee (2 \in A)$

$2 \in (A \cup B) \Leftrightarrow (2 \in A) \vee (2 \in B)$

$$\begin{aligned}
[2 \notin (B - A)] \wedge [2 \in (A \cup B)] &\Leftrightarrow [(2 \notin B) \vee (2 \in A)] \wedge [(2 \in A) \vee (2 \in B)] \\
&\Leftrightarrow [(2 \notin B) \wedge (2 \in B)] \vee (2 \in A) \\
&\Leftrightarrow F \vee (2 \in A) \\
&\Leftrightarrow 2 \in A
\end{aligned}$$

Then, $A = \{2,3,4,5\}$ and $B = \{1,3,4,5\}$

3.2 Relations

The notion of relation is omnipresent, in mathematics as in everyday life. The intuitive idea is to understand the fact that a certain link exists or not between two or more objects.

The concept of relation finds a precise characterization in a mathematical context, the Cartesian product operation offering in this respect a frame both propitious and fertile.

3.2.1 Binary Relations

Binary relations are an excellent way for capturing certain structures that appear in computer science.

Definition (Binary Relation)

A **binary relation** over a nonemty set A is a predicate R that can be applied to ordered pairs (x, y) of elements x and y given from A.

The representing **graph** of a relation in A is a graph $G \subseteq A \times A$ which consists of all the pairs (x, y) such that the relation between two elements x and y is true. Conversely, if we are given an arbitrary graph $G \subseteq A \times A$, then G defines a relation in A, namely the relation R is true if and only if $(x, y) \in G$.

Notation for Binary Relations

Let R be a binary relation in A . Then

$$\boxed{x\ R\ y \Longleftrightarrow (x, y) \in G}$$

Examples:

1. "x is greater than y ".
2. "x and y have the same absolute value".
3. "$x^2 + y^2 = 1$".
4. $A \subset B$.

Example: Suppose $A = \{1, 2, 3, 4\}$. We give the graph $G \subseteq A \times A$ of the following relation:

$\forall\ x,\ y \in A: \quad x\ R\ y \quad \Longleftrightarrow \quad x < y.$

Then, $G = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

Example: A relation R is defined on $\mathbb{R}$ by:

$\forall\ x,\ y \in \mathbb{R}: \quad x\ R\ y \quad \Longleftrightarrow \quad xy^3 - x^3y = 6$

We show that $1\ R\ 2$, because $1 \times 2^3 - 1^3 \times 2 = 6$

Remark: If R is a binary relation over A and it does not hold for the pair (x, y), then $\overline{x\ R\ y}$.

Examples: $3 \neq 4, \mathbb{R} \not\subseteq \mathbb{Z}, 4 \nleq 3$.

Properties of a relation

Let R be a binary relation in A.

- R is **reflexive** iff $\forall x \in A: \ x \ R \ x$.
- R is **symmetric** iff $\forall x, y \in A: x \ R \ y \Rightarrow y \ R \ x$.
- R is **anti-symmetric** iff $\forall x, y \in A: (x \ R \ y) \wedge (y \ R \ x) \Longrightarrow x = y$.
- R is **transitive** iff $\forall x, y, z \in A: (x \ R \ y) \wedge (y \ R \ z) \Longrightarrow x \ R \ z$.

Example: Let $A = \{1, 2, 3\}$ and consider three relations R, T, S on A:
$G_R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$.
$G_S = \{(1,1), (1,3), (2,2), (3,2)\}$.
$G_T = \{(1,2), (1,3), (2,3)\}$.

- R is reflexive, symmetric, and transitive, but not anti-symmetric because $(1 \ R \ 2) \wedge (2 \ R \ 1)$ but $1 \neq 2$.
- S is anti-symmetric, but not reflexive because $\left(\overline{3 \ R \ 3}\right)$, not symmetric $(1 \ R \ 3 \ )$ but $\overline{(3 \ R \ 1)}$, and not transitive $(1 \ R \ 3) \wedge (3 \ R \ 2)$ but $\left(\overline{1 \ R \ 2}\right)$.
- T is anti-symmetric and transitive, but not reflexive $\left(\overline{1 \ R \ 1}\right)$ and not symmetric $(1 \ R \ 2 \ )$ but $\overline{(2 \ R \ 1)}$.

3.2.2 Equivalence Relation

Definition (Equivalence Relation)

An **equivalence relation** is a relation that is *reflexive, symmetric and transitive.*

Examples:

1. The "*equal-to*" relation, "$=$", on $\mathbb{R}$ is an equivalence relation.
2. The "*less- than -or- equal to* " relation, "$\leqslant$", on $\mathbb{R}$ is not an equivalence relation because it is not symmetric. For example: $1 \leqslant 2$ but $2 \nleqslant 1$.
3. The "*strictly-less-than*" relation, "$<$ ", on $\mathbb{R}$ is not an equivalence relation because it is not reflexive. For example: $1 \nless 1$.
4. The "*Line Parallel relation*", "$\|$" , is an equivalence relation.

5. The "*Perpendicular Lines*" relation, "$\perp$", is symmetric but neither reflexive nor transitive.

6. The "subset" relation "$\subseteq$", on $P(E)$ such that $E = \{1, 2, 3\}$, is not an equivalence relation because it is not symmetric. For example: $\{1\} \subseteq \{1, 2\}$ but $\{1, 2\} \nsubseteq \{1\}$.

Exercise: Let A be the set of all triangles in a plane with R a relation in A given by

$$\forall\, T_1,\ T_2 \ \in A: \ \ T_1\ R\ T_2 \iff T_1 \text{ is congruent to } T_2.$$

Show that R is an equivalence relation.

Solution:

1. R is reflexive, since every triangle is congruent to itself.

2. $T_1\ R\ T_2 \Longrightarrow T_1$ is congruent to $T_2 \Longrightarrow T_2$ is congruent to $T_1 \Longrightarrow T_2\ R\ T_1$. Hence, R is symmetric.

3. $(T_1 R\ T_2) \wedge (T_2 R\ T_3) \Longrightarrow T_1$ is congruent to T_2 and T_2 is congruent to $T_3 \Longrightarrow T_1$ is congruent to T_3. Hence, R is transitive.

Therefore, R is an equivalence relation.

Exercise: Consider the binary relation R defined over the set $\mathbb{Z}$:

$$\forall\, x,\ y \ \in \mathbb{Z}: \ \ x\ R\ y \iff x + y \text{ is even.}$$

Show that R is an equivalence relation.

Solution:

(a) R is **reflexive** $\Leftrightarrow \forall x \in \mathbb{Z}: \ x\ R\ x$.

1. Let $x \in \mathbb{Z}$, the sum $x + x$ can be written as $2k$ for some integer k (namely, x), so $x + x$ is even. Then $x\ R\ x$ holds, as required.

(b) R is **symmetric** $\Leftrightarrow \ \forall x, y \in \mathbb{Z}: x\ R\ y \Rightarrow y\ R\ x$.

Let $x, y \in \mathbb{Z}$,

$$\begin{aligned} x\ R\ y \quad &\Rightarrow \quad \exists k \in \mathbb{Z},\ x + y = 2k \\ &\Rightarrow \quad y + x = x + y = 2k \quad \text{(by Commutative Property of Addition)} \\ &\Rightarrow \quad y + x \text{ is even} \\ &\Rightarrow \quad y\ R\ x, \text{ as required} \end{aligned}$$

(c) R is **transitive** $\Leftrightarrow\ \forall x, y, z \in \mathbb{Z} : (x\ R\ y) \wedge (y\ R\ z) \Longrightarrow x\ R\ z$.

Let $x, y, z \in \mathbb{Z}$,

$$\begin{array}{rcc}
(x\ R\ y) \wedge (y\ R\ z) & \Rightarrow & \exists\ k, k' \in \mathbb{Z},\ (x+y=2k) \wedge \left(y+z=2k'\right) \\
& \Rightarrow & x+y+y+z = 2k+2k' \\
& \Rightarrow & x+z = 2k+2k'-2y = 2k'' /\ k'' = k+k'+y \\
& \Rightarrow & x+z \text{ is even} \\
& \Rightarrow & x\ R\ z,\ \text{holds, as required.}
\end{array}$$

Therefore, R is an equivalence relation.

3.2.3 Equivalences and Partitions

Definition (Equivalence Relation)

Given a partition $A_1, A_2, A_3, \ldots$ of a set A, two elements x and y of A are said to be **equivalent**, with respect to that partition, if they belong to the same subset specified by the partition.

Example: The days of the year are partitioned by seven disjoint sets given by the weekday names of the days. For instance, August 1, 1966, and June 30, 2003, are equivalent in this context since they both belong to the subset called "Monday."

Example: Assuming two words of the English language to be equivalent if they each possess the same number of vowels is an equivalence relation on the set of all words.

Definition (Equivalence Classes)

Given an equivalence relation R over a set A, for any $x \in A$, the equivalence class of x is the set

$$[x] = \{y,\ x\ R\ y\}$$

$[x]$ is the set of all elements of A that are related to x by relation R.

Property: If R is an equivalence relation over A, then every $x \in A$ belongs to exactly one equivalence class.

Theorem: If R is an equivalence relation on a set A, then the collection of its equivalence classes is a partition of A. Conversely, if P is a partition of A, then the relation defined by

$$x\ R\ y \Leftrightarrow \exists\ S \in P : x, y \in S$$

is an equivalence relation, and its equivalence classes are the elements of the partition.

Exercise: Provide a proof.

3.2.4 Order Relation

The notion of order relation on a set aims to define the intuitive idea that an object "precedes" another, "come before" it, according to a certain criterion of ordering, of disposition of the objects in question.

Definition (Order Relation)

A binary relation R on a set A is called an **order relation** if it is *reflexive, anti-symmetric, and transitive.*

Examples:

1. The "*less-than-or-equal-to*" relation on the set of integers $\mathbb{Z}$ is an order relation.
2. The "*strictly-less-than*" and "*proper-subset*" relations are not order relation because they are not reflexive.

Definition (Total Ordering Relation)

An **order relation** R on A is called a **total ordering** if it satisfies one additional proposition:

$$\boxed{\forall x \in A, \forall y \in A: \ (x\ R\ y) \vee (y\ R\ x)}$$

Examples:

1. The relation $(\mathbb{R}, \leqslant)$ is a total order relation.
2. The relation $(P(E), \subseteq)$ is not a total order relation because $\exists\ A, B \in P(E) : (A \nsubseteq B) \wedge (B \nsubseteq A)$.

Example: Show that the relation "Divides" defined on $\mathbb{N}^*$ is an order relation.

Solution:

1) R is **reflexive** $\Leftrightarrow \forall x \in \mathbb{N}^* :\ x\ R\ x$

We have x divides x, $\forall x \in \mathbb{N}^*$. Therefore, relation "*Divides*" is reflexive.

2) R is **anti-symmetric** $\Leftrightarrow\ \forall x, y \in \mathbb{N}^* : (x\ R\ y) \wedge (y\ R\ x) \Longrightarrow x = y$

Let $x, y \in \mathbb{N}^*$,

$$\begin{array}{lll}(x\ R\ y) \wedge (y\ R\ x) & \Rightarrow & (x \text{ divides } y) \text{ and } (y \text{ divides } x) \\ & \Rightarrow & \exists\, k, k' \in \mathbb{N}^*, \left(y = k' x\right) \wedge (\ x = ky) \\ & \Rightarrow & x = kk' x \\ & \Rightarrow & kk' = 1 \\ & \Rightarrow & k = k' = 1 \in \mathbb{N}^* \\ & \Rightarrow & x = y\end{array}$$

So, the relation is anti-symmetric.

3) R is **transitive** $\Leftrightarrow\ \forall x, y, z \in \mathbb{N}^* : (x\ R\ y) \wedge (y\ R\ z) \Longrightarrow x\ R\ z$.
Let $x, y, z \in \mathbb{N}^*$,

$$\begin{array}{lll}(x\ R\ y) \wedge (y\ R\ z) & \Rightarrow & \exists\, k, k' \in \mathbb{N}^*, (y = k\ x) \wedge \left(\ z = k' y\right) \\ & \Rightarrow & z = k' k\ x \\ & \Rightarrow & z = k''\ x\ /\ \ k'' = k' k \in \mathbb{N}^* \\ & \Rightarrow & x \text{ divides } z \Rightarrow x\ R\ z\end{array}$$

Hence, the relation is transitive.

Thus, the relation R being reflexive, anti-symmetric and transitive, the relation “*divides*” is an order relation.

3.3 Applications

3.3.1 Functional Relation

The concept of a *function* is one of the most basic mathematical ideas and enters into almost every mathematical discussion.We focus on the concept of *Functional relation* which is called an *application*. We will not study derivatives or integrals, but rather the notions of *injective and surjective applications*, how to compose *applications*, and when they are *invertible*.

Definition (Function)

Let E and F be sets. A **function** is a relation from a set E to another set F, denoted by $f : E \to F$, that *every element* $x \in E$ assigns *at most a unique element* $y \in F$ satisfying $x\ f\ y$. To indicate this relation between x and y we usually write $y = f(x)$.

$$\boxed{x\ f\ y \Longleftrightarrow y = f(x)}$$

We say that:

- y is the **image** of x (under f).
- x is the **pre-image** of y (under f).
- f **maps** x onto y, and symbolize this statement by $x \overset{f}{\longmapsto} y$.
- E is the **starting set** of f.
- F is the **arrival set** or **codomain** of f.
- G_f is the **graph** of the function f, given by: $G_f = \{(x, y) \in E \times F \ / \ y = f(x)\}$

Note that

$$f: \begin{array}{ccc} E & \longrightarrow & F \\ x & \longmapsto & y \end{array}$$

Definition (Domain Definition)
The **Domain** D_f of a function f of E in F is the set of elements $x \in E$ satisfying: there is one and only one element $y \in F$ such that $y = f(x)$.
Definition (Range)
We call **range** of a function f the subset of F with *preimages*.
Definition (Application)
An **application** f is a function of E in F whose domain definition D_f is equal to E.
Definition (Application)
An **application** from a set E to a another set F is a relation which to *every element* $x \in E$ assigns a *unique element* $y \in F$. Formally, using predicate logic:

$$\boxed{f \ \textbf{Application} \Leftrightarrow \begin{cases} 1) \quad \forall\, x \in E,\ \exists\, y \in F: \quad y = f(x) \\ 2) \quad \forall x_1,\ x_2 \in E: x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \end{cases}}$$

we can write also,

$$\boxed{f \ \textbf{Application} \Leftrightarrow \{\forall\, x \in E,\ \exists!\, y \in F: \quad y = f(x)}$$

An **application** (or function) is a triplet $f = (E, F, R)$, where E and F are two sets and $G_f \subseteq E \times F$ is a functional relation (See Figure 3.9).

Example: Consider the assignment rule $f : E = \{1, 2, 3, 4\} \to F = \{x, y, z\}$ which is defined by:

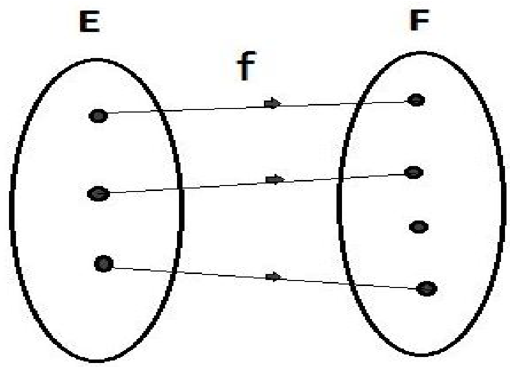


Figure 3.9

1. $G = \{(1,x),(2,y),(3,z),(4,x)\}$
2. $G = \{(1,x),(2,x),(3,x),(4,y)\}$
3. $G = \{(1,z),(2,y),(3,x)\}$
4. $G = \{(1,y),(2,x),(3,y),(3,z),(4,x)\}$

The first two relations are applications and the third relation is function with D_f is $\{1,2,3\}$ but not the last one.

3.3.2 Equality - Extension - Restriction

Definition (The equality of applications)
Two applications $f = (A,B,R)$ and $g = (C,D,S)$ are called equal if and only if they have the same domain $A = C$, the same codomain $B = D$ and the same graphic $G_f = Gg$. If $f, g : A \longrightarrow B$, the **equality** $f = g$ is equivalent to $f(x) = g(x)$, $\forall x \in A$, that is to say:

$$\boxed{f = g \Longleftrightarrow \forall x \in A,\ f(x) = g(x)}$$

Definition (Extension of an application-Restriction of an application)

Let $f : X \to Y$ be an application and A and B be sets such that $X \subseteq A$ and $Y \subseteq B$. An **extension** of f to A is an application $g : A \to B$ such that $f(x) = g(x)$ for all $x \in X$. Alternatively, g is an extension of f to A if f is the **restriction** of g to X.

3.3.3 Image and Inverse Image of a Subsets

Often in mathematics, particularly in analysis and topology, one is interested in finding the set of image points or inverse image of an application acting on a given set, which brings us to the two following definitions that are waiting to be understood.

Definition (Image of a Subset)

Let $f : E \longrightarrow F$ and consider the subset $A \subset E$. The **image** of the subset A under f , which we write $f(A)$, is the subset of F that consists of the *images* of the elements of A (See Figure 3.10):

$$\boxed{f(A) = \{f(x), \quad x \in A\}}$$

$$\boxed{y \in f(A) \Longleftrightarrow \exists\, x \in A, y = f(x)}$$

Definition (Inverse Image of a Subset)

Let $f : E \longrightarrow F$ and consider the subset $B \subset F$. The **inverse image** of the subset B under f, which we write $f^{-1}(B)$ is the subset of E that consists of the *pre-images* of elements in B (See Figure 3.10)

$$\boxed{f^{-1}(B) = \{x,\ f(x) \in B\}}$$

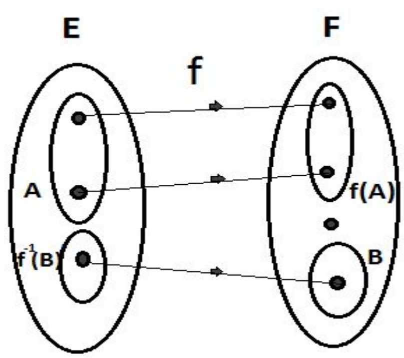


Figure 3.10

Example: Let $E = \{1, 2, 3, 4\}$ and $F = \{a, b, c\}$ and define an application $f : E \longrightarrow F$ such that $f\ (1) = f(2) = a$, $f(3) = f(4) = c$. Let $A \subset E$, $A = \{1, 2, 3\}$. Then $f(A) = \{a, c\}$. Also for example $f^{-1}(\{b\}) = \emptyset$, $f^{-1}(\{a, c\}) = A$, $f^{-1}(\{b, c\}) = \{3\}$.

Properties

Given an application $f : E \longrightarrow F$ where A, B, are subsets of E and C, D, are subsets of F, we have the following properties. Notice how the inverse image always preserves unions and intersections, although not always true for the image of an application. Then the images of intersections and unions satisfy:

1. $f(A \cap B) \subseteq f(A) \cap f(B)$.
2. $f(A \cup B) = f(A) \cup f(B)$.
3. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
4. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
5. $A \subseteq B \Longrightarrow f(A) \subseteq f(B)$.
6. $C \subseteq D \Longrightarrow f^{-1}(C) \subseteq f^{-1}(D)$.
7. $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$.

Exercise: Let $f(x) = 1 + x^2$. Find the following:

1. $f(\{-1, 1\})$.
2. $f([-2, 2])$.
3. $f([-2, 3])$.
4. $f^{-1}(\{1, 5, 10\})$.
5. $f^{-1}([0, 1])$.
6. $f^{-1}([2, 5])$.

Solution:

1. $f(\{-1, 1\}) = \{2\}$.
2. $f([-2, 2]) = f([-2, 0] \cup [0, 2]) = f([-2, 0]) \cup f([0, 2]) = [1, 5]$.
3. $f([-2, 3]) = f([-2, 0] \cup [0, 3]) = f([-2, 0]) \cup f([0, 3]) = [1, 10]$.
4. $f^{-1}(\{1, 5, 10\}) = \{0, 2, -2, 3, -3\}$.
5. $f^{-1}([0, 1]) = \{0\}$.
6. $f^{-1}([2, 5]) = [-2, -1] \cup [1, 2]$.

3.3.4 Injective, Surjective and Bijective Applications

Definition (Injective)

An **injective** application (or **one-to-one** application) $f : E \longrightarrow F$, is an application for which *every element of the range of the application corresponds to exactly one element of the domain.* Formally:

$$f \ \textbf{Injective} \iff \begin{cases} \forall x_1,\ x_2 \in E : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\ \text{or} \\ \forall x_1,\ x_2 \in E : f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \end{cases}$$

In words, this says that all elements in the domain of f have different images (See Figure 3.11)

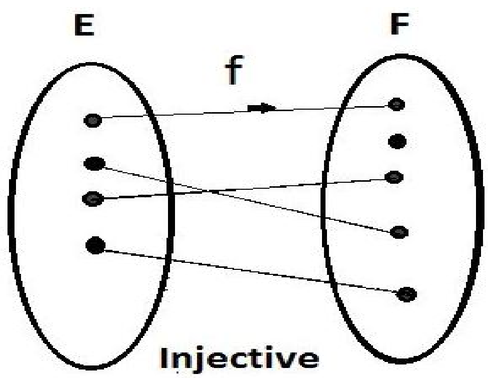


Figure 3.11

Example. Consider an application $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 4x - 1$. We want to know whether each element of $\mathbb{R}$ has a different image. In fact, this function is a line, so one may "see" that two distinct elements have distinct images, but let us try a proof of this.

$$\begin{aligned} f(x_1) = f(x_2) &\implies 4x_1 - 1 = 4x_2 - 1 \\ &\implies 4x_1 = 4x_2 \\ &\implies x_1 = x_2 \end{aligned}$$

Therefore f is injective.

Example. Consider an application $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(x) = x^2$. A property of injectivity of g is not true by providing an example where it does not hold. The two elements $x_1 = 1$ and $x_2 = -1$ are both sent to $g(x_1) = g(x_2) = 1$.

The other definition that always comes in pair with that of injective is that of surjective.

Definition (Surjective)
An application $f : E \longrightarrow F$ is **surjective** (or **onto**) if and only if *for every element* $y \in F$*, there is an element* $x \in E$ with $y = f(x)$:

$$\boxed{\forall y \in F, \exists\, x \in E : y = f(x)}$$

In words, each element in the co-domain of f has a pre-image (See Figure 3.12)

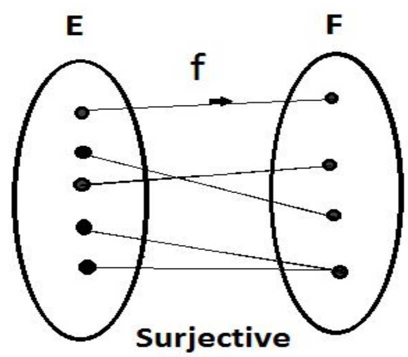


Figure 3.12

Example. Consider again $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 4x - 1$. We want to know whether each element of $\mathbb{R}$ has a preimage.

$$\begin{aligned} f(x) = y \quad &\Longrightarrow \quad 4x - 1 = y \\ &\Longrightarrow \quad 4x = y + 1 \\ &\Longrightarrow \quad x = \tfrac{y+1}{4} \in \mathbb{R} \end{aligned}$$

Therefore f is surjective.

Example. Consider again $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(x) = x^2$. A property of surjectivity of g is not even true by providing an example where it does not hold. If $y = -1$, there is no $x \in \mathbb{R}$ such that $g(x) = x^2 = -1$.

Exercise: The function f is defined by: $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2 - 6x$

1. Give an example to show that f is not injective.

2. Give an example to show that f is not surjective.

Solution:

1. $f(6) = f(0) = 0$ but $6 \neq 0$, therefore the application is not injective.

2. $f(x) = x^2 - 6x = (x-3)^2 - 9$

$$\text{let } y = -10 \quad \text{then} \quad f(x) = -10 \implies (x-3)^2 - 9 = -10$$
$$\implies (x-3)^2 = -1$$

There is no real number, x such that $f(x) = -10$ the application is not surjective.

Or the range of the application is $y \geq 2$. The range of the function is not $\mathbb{R}$ (the codomain), therefore the application is not surjective.

We next combine the definitions of an application which is injective and surjective, to get:

Definition 1 (Bijective)

An application $f : E \longrightarrow F$ is **bijective** if and only if *it is both injective and surjective* (See Figure 3.13)

Definition 2 (Bijective)

An application $f : E \longrightarrow F$ is **bijective** if and only if *for every element* $y \in F$, *there is a unique element* $x \in E$ with $y = f(x)$:

$$\boxed{\forall y \in F, \exists!\ x \in E : y = f(x)}$$

Example: Consider the application $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 4x - 1$, which we have just studied in two previous examples. We know it is both injective and surjective, therefore it is a bijection.

Bijections have a special feature: they are **invertible**, formally:

Definition 1 (Inverse Application)

Let $f : E \longrightarrow F$ be a bijection. Then the **inverse application** of f , $f^{-1} : F \longrightarrow E$ is defined elementwise by: $f^{-1}(y)$ is the *unique element* $x \in E$ such that $f(x) = y$. We say that f is **invertible**.

Example: Let us consider again at our two previous examples, namely, $f(x) = 4x - 1$ and $g(x) = x^2$. Then, the application g is not a bijection, so it cannot have an inverse. Now f is an application bijective, so we can compute its inverse.

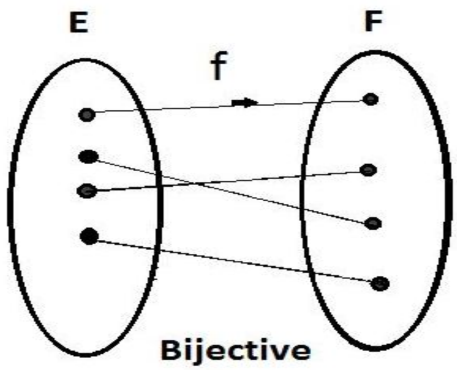


Figure 3.13

$$
\begin{aligned}
y = f(x) &\iff y = 4x - 1 \\
&\iff y + 1 = 4x \\
&\iff x = \tfrac{y+1}{4} \\
&\iff f^{-1}(y) = \tfrac{y+1}{4}
\end{aligned}
$$

Property: Let $f : E \longrightarrow F$ be a bijective application , then $f^{-1} : F \longrightarrow E$ is a bijective application.

3.3.5 Examples of Applications

- **Identity Application**. Let A be a set; by the identity application on A we mean the application $I_A : A \to A$ given by

$$\boxed{I_A(x) = x}$$

  * I_A is injective, $I_A(x) = I_A(y) \Longrightarrow x = y$ ($I_A(x) = x$ and $I_A(y) = y$); thus the injection holds.
  * I_A is surjective because, obviously, the *range* of I_A is A.
  * Thus I_A is bijective.

- **Constant Application**. Let A and B be sets, and let b be an element

  of B. By the constant application f_b we mean the application f_b : $A \to B$ given by:

$$\boxed{f_b(x) = b, \forall x \in A}$$

- **Characteristic Application.** The characteristic application of a set is used to solve some difficult problems of set theory found in undergraduate studies. Let's consider $A \subset E \neq \emptyset$ (a universal set), then $f_A : E \to \{0, 1\}$, where the application

$$f_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is called the characteristic application of the set A.

3.3.6 Operations on Applications

Given two applications, it may be possible to combine them in different ways to create more complicated applications. If the domains and codomains of the two applications agree and if the codomain supports arithmetic, we may define arithmetic operations on the applications by point-wise operations on their images.

Arithmetic Operations on Applications

Let $f : E \to F$ and $g : E \to F$ be two applications sharing a common domain E and let α be a real number. Then $f + g$, $f - g$, $\alpha \cdot f$, $f{\cdot}g$, and f/g $(g(x) \neq 0)$ denote the following applications from E to F :

a) $(f + g)(x) = f(x) + g(x)$.
b) $(f - g)(x) = f(x) - g(x)$.
c) $(c{\cdot}f)(x) = c \cdot f(x)$.
d) $(f{\cdot}g)(x) = f(x) \cdot g(x)$.
e) $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$.

Exercise: Find counter-examples to each of these statements for $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$:
(a) If f and g are surjective, then $(f + g)$ is surjective.
Suppose $f(x) = x$ and $g(x) = -x$. Then $(f + g)(x) = x - x = 0$.
(b) If f and g are surjective, then $f{\cdot}g$ is surjective.

The same $f(x) = x$ and $g(x) = -x$ from above work; $(f{\cdot}g)(x) = -x^2$, which is not surjective.

Composition of Applications

In addition to arithmetic operations on applications, there is another operation called composition of applications which is more set-theoretic or algebraic in nature. A composite of two applications is satisfied if the codomain of the first application agrees with the domain of the second.

If $f : E \to F$ and $g : F \to G$ are applications in which the codomain of f equals the domain of g, then the assignment $h(x) = g(f(x))$ defines an application $h : E \to G$. For, given any $x \in E$, there is a unique $y \in F$ such that $y = f(x)$, since f is an application. Similarly, since g is an application, $g(f(x))$ is a unique image in G. Thus each element x from E yields a unique image $z = g(f(x))$ in G, guaranteeing that h is an application from E into G. This legitimizes the following definition.

Definition (Composite Applications)

If $f : E \to F$ and $g : F \to G$, then the composite application f followed by g is the application $g \circ f$ such that:

$$\begin{array}{llcll} g \circ f: & A & \to & C & \\ & x & \mapsto & (g \circ f)(x) & = g(f(x)) \end{array}$$

Example: If $f(x) = -4x + 9$ and $g(x) = 2x - 7$, find $(f \circ g)(x)$ and $(g \circ f)(x)$

$$\begin{array}{rcc} (f \circ g)(x) & = & f(g(x)) \\ & = & -4g(x) + 9 \\ & = & -4(2x - 7) + 9 \\ & = & -8x + 28 + 9 \\ & = & -8x + 37 \end{array}$$

Thus, $(f \circ g)(x) = -8x + 37$.

$$\begin{array}{rcc} (g \circ f)(x) & = & g(f(x)) \\ & = & 2f(x) - 7 \\ & = & 2(-4x + 9) - 7 \\ & = & -8x + 18 - 7 \\ & = & -8x + 11 \end{array}$$

Thus, $(g \circ f)(x) = -8x + 37$.

We remark that $(f \circ g)(x)$ and $(g \circ f)(x)$ produced different answers.

Properties: Suppose f, g, and h are application that can be composed in the order given.

1. Composition is not Commutative: $g \circ f \neq f \circ g$.

2. If f and g are both injective applications, then so is $g \circ f$.

3. If f and g are both surjective applications, then so is $g \circ f$.

4. If f and g are both bijective applications, then so is $g \circ f$.

5. Composition is Associative: $(h \circ g) \circ f = h \circ (g \circ f)$.

Definition 2 (Inverse Applications)

If $f : E \to F$ and $g : F \to E$, then f and g are inverse applications of one another relative to composition iff

$$g \circ f = I_E \text{ and } f \circ g = I_F.$$

Example: Show that the application $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{x-1}{2}$ is an inverse for the application $g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$.

$$g(f(x)) = \frac{f(x)-1}{2} = \frac{2x+1-1}{2} = x$$

$$f(g(x)) = 2g(x) + 1 = 2\left(\frac{x-1}{2}\right) + 1 = x$$

Thus g is f's inverse.

Properties: Let $f : E \to F$ and $g : F \to G$

1. If f has an **inverse**, then it is unique.

2. The composition $g \circ f$ of two invertible applications f and g is invertible. Moreove , the composition of the inverses in the reverse order

$$\boxed{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}$$

Example: Let an application $f : E \to F$.

Determine the inverse application for $f(x) = \frac{x}{x+1}$. Assume that f is defined for as inclusive a set of real numbers as possible and that the codomain of f is its range.

The equation $y = \frac{x}{x+1}$ defines an application on $E = \mathbb{R} - \{-1\}$.

We will check its codomain after we determine which values y can be. Solving $y = \frac{x}{x+1}$ for x, we get the following:

$$\begin{aligned} y = \frac{x}{x+1} &\Rightarrow y(x+1) = x \\ &\Rightarrow yx + y = x \\ &\Rightarrow yx - x = -y \\ &\Rightarrow x = \frac{y}{1-y} = g(y) \end{aligned}$$

Since there are x-values for all y except $y = 1$, our domain for g and our codomain for f must be taken to be $F = \mathbb{R} - \{1\}$.

For these x- and y-values the above solution process is reversible. The inverse application is therefore given by $f^{-1}(y) = \frac{y}{1-y}$.

3.4 The Inverse Trigonometric Application

In this section, we concern ourselves with finding inverses of the (circular) trigonometric applications. Our immediate problem is that, owing to their periodic nature, none of the circular applications is *injective*. To remedy this, we restrict the domains of the circular applications to obtain an *injective* application.

3.4.1 Arccosine Application

We first consider $f(x) = \cos(x)$. Choosing the interval $[0, \pi]$ allows us to keep the range as $[-1, 1]$ as well as the property of being *bijective*.

Recall from Subsection 2.3.4 that the inverse of an application f is typically denoted f^{-1}. For this reason,we can use the notation $f^{-1}(x) = \cos^{-1}(x)$ for the inverse of $f(x) = \cos(x)$ (See Figures 3.14–3.15)

Remark: It is far too easy to confuse $\cos^{-1}(x)$ with $\frac{1}{\cos(x)}$ so we will not use this notation in our text.

Notation: We use the notation $f^{-1}(x) = \arccos(x)$, read "**arc-cosine** of x".

Formally:

$$\begin{array}{rccl} f^{-1}: & [-1, 1] & \rightarrow & [0, \pi] \\ & x & \mapsto & f^{-1}(x) \quad = \arccos(x) \end{array}$$

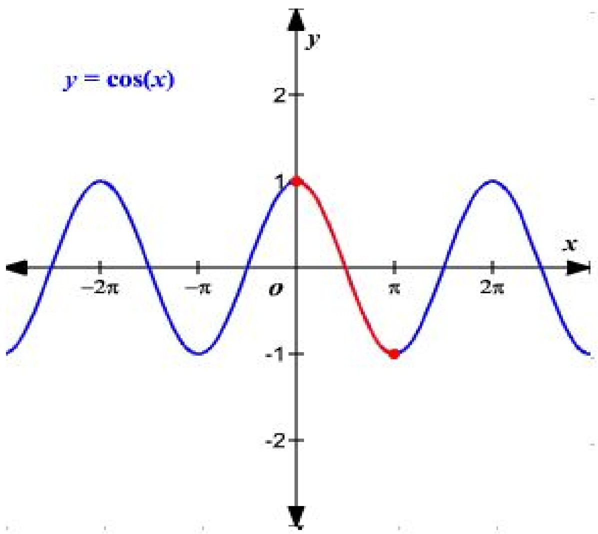
We list some important facts about the arccosine applications in the following properties.

Properties

$\arccos(x) = y$ if and only if $y \in [0, \pi]$ and $\cos(y) = x$.

$\cos(\arccos(x)) = x$ provided $x \in [-1, 1]$.

$\arccos(\cos(x)) = x$ provided $x \in [0, \pi]$.



Extract from https://www.varsitytutors.com/hotmath

Figure 3.14

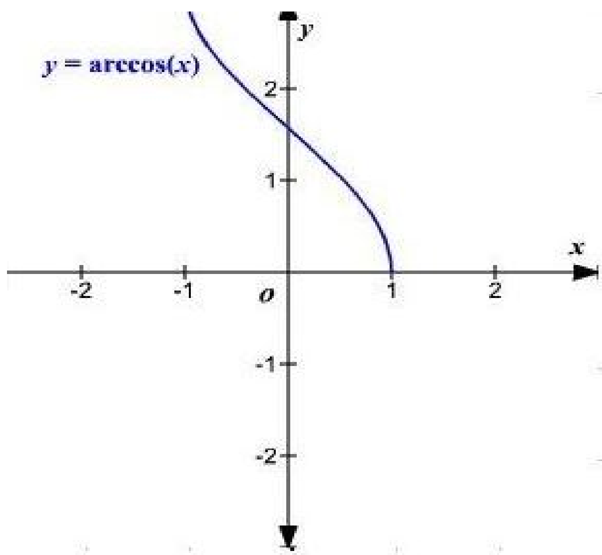


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Figure 3.15

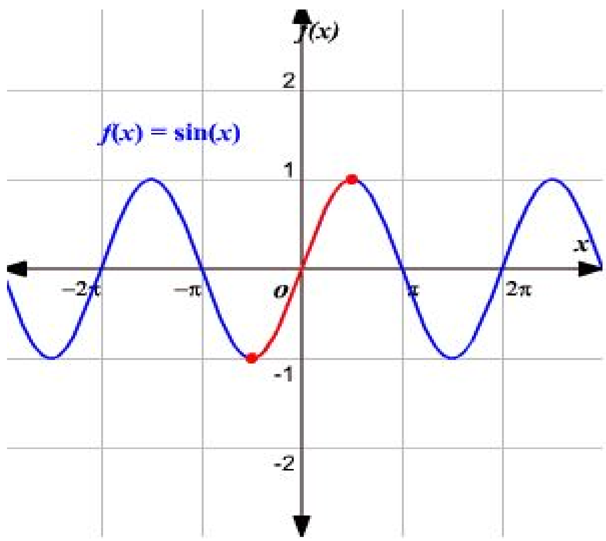

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Figure 3.16

3.4.2 Arcsine Application

We restrict $f(x) = \sin(x)$ in a similar manner, although the interval of choice is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (See Figure 3.16)

It should be no surprise that we call $f^{-1}(x) = \arcsin(x)$, which is read "**arc-sine** of x"(See Figure 3.17)

Formally:

$$\begin{array}{rcll} f^{-1}: \ [-1,1] & \to & \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & \\ x & \mapsto & f^{-1}(x) & = \arcsin(x) \end{array}$$

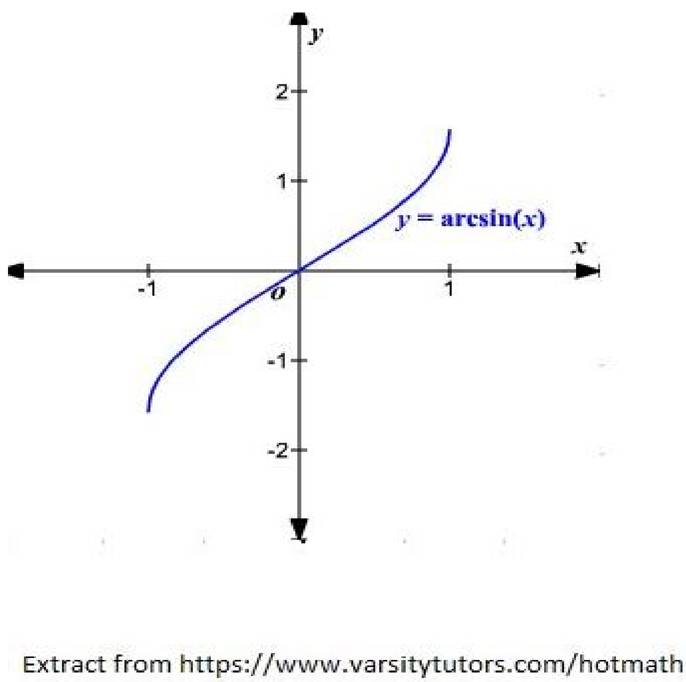

Figure 3.17

We list some important facts about the arcsine applications in the following properties.

Properties

$\arcsin(x) = y$ if and only if $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin(y) = x$.

$\sin(\arcsin(x)) = x$ provided $x \in [-1, 1]$.
$\arcsin(\sin(x)) = x$ provided $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Exercise: Find the exact values of the following.

1) $\arccos(\frac{1}{2})$	5) $\arcsin(\frac{\sqrt{2}}{2})$
2) $\arccos(-\frac{\sqrt{2}}{2})$	6) $\arcsin(-\frac{1}{2})$
3) $\arccos(\cos\left(\frac{\pi}{6}\right))$	7) $\arccos(\cos(\frac{11\pi}{6}))$
4) $\cos\left(\arccos(-\frac{3}{5})\right)$	8) $\sin\left(\arccos(-\frac{3}{5})\right)$

Solution:

1. To find $\arccos(\frac{1}{2})$, we need to find the real number y (or, equivalently, an angle measuring y radians) which verifies $y \in [0, \pi]$ and with $\cos(y) = \frac{1}{2}$. We know $y = \frac{\pi}{3}$ meets these criteria, so $\arccos(\frac{1}{2}) = \frac{\pi}{3}$.

2. The number $y = \arccos(-\frac{\sqrt{2}}{2}) \in [0, \pi]$ with $\cos(y) = -\frac{\sqrt{2}}{2}$. Our answer is $y = \frac{3\pi}{4}$.

3. Since $\frac{\pi}{6} \in [0, \pi]$, we could simply refer to the properties of arccosine applications to get $\arccos(\cos\left(\frac{\pi}{6}\right)) = \frac{\pi}{6}$.

4. One way to simplify $\cos\left(\arccos(-\frac{3}{5})\right)$ is to use the properties of arccosine applications directly. Since $-\frac{3}{5} \in [-1, 1]$, we have $\cos\left(\arccos(-\frac{3}{5})\right) = -\frac{3}{5}$.

5. The value of $\arcsin(\frac{\sqrt{2}}{2})$ is a real number $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(y) = \frac{\sqrt{2}}{2}$. The number we seek is $y = \frac{\pi}{4}$. Hence, $\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.

6. To find $\arcsin(-\frac{1}{2})$, we seek the number $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(y) = -\frac{1}{2}$. The answer is $y = -\frac{\pi}{6}$ so that $\arcsin(-\frac{1}{2}) = -\frac{\pi}{6}$.

7. Since $\frac{11\pi}{6}$ does not fall between 0 and π, the properties of the arcsine applications does not apply. We are forced to work through from the inside out starting with $\arccos(\cos(\frac{11\pi}{6})) = \arccos(\frac{\sqrt{3}}{2})$. We know $\arccos(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$. Hence, $\arccos(\cos(\frac{11\pi}{6})) = \frac{\pi}{6}$.

8. As in the previous question, we let $y = \arccos(-\frac{3}{5})$ so that $\cos y = -\frac{3}{5}$ for $y \in [0, \pi]$. Since $\cos y < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < y < \pi$. In terms of y, then, we need to find $\sin\left(\arccos(-\frac{3}{5})\right) = \sin y$. Using the Pythagorean Identity $\cos^2 y + \sin^2 y = 1$, we get $\left(-\frac{3}{5}\right)^2 + \sin^2 y = 1$ or $\sin y = \pm\frac{4}{5}$. We choose $\sin y = \frac{4}{5}$. Hence, $\sin\left(\arccos(-\frac{3}{5})\right) = \frac{4}{5}$.

The next pair of application we wish to discuss are the inverses of tangent and cotangent, which are named arctangent and arccotangent, respectively.

3.4.3 Arctangent Application

We restrict $f(x) = \tan(x)$ to its fundamental cycle on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ to obtain $f^{-1}(x) = \arctan(x)$. Among other things, note that the *vertical asymptotes* $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ of the graph of $f(x) = \tan(x)$ become the horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(x) = \arctan(x)$. We show these graphs on Figure 18.

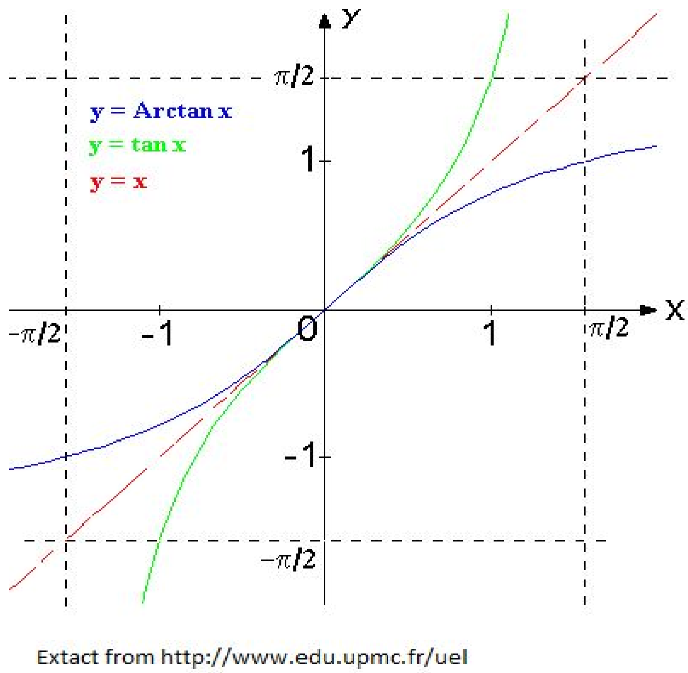

Figure 3.18

We list some of the basic properties of the arctangent application.

Properties

$\arctan(x) = y$ if and only if $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $\tan(y) = x$.

$\tan(\arctan(x)) = x$ provided $x \in \mathbb{R}$.

$\arctan(\tan(x)) = x$ provided $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

3.4.4 Arccotangent Application

We restrict $f(x) = \cot(x)$ to its fundamental cycle on $]0, \pi[$ to obtain $f^{-1}(x) = arccot(x)$. Once again, the *vertical asymptotes* $x = 0$ and $x = \pi$ of the graph of $f(x) = \cot(x)$ become the *horizontal asymptotes* $y = 0$ and $y = \pi$ of the graph of $f^{-1}(x) = arccot(x)$. We show these graphs on Figure 19.

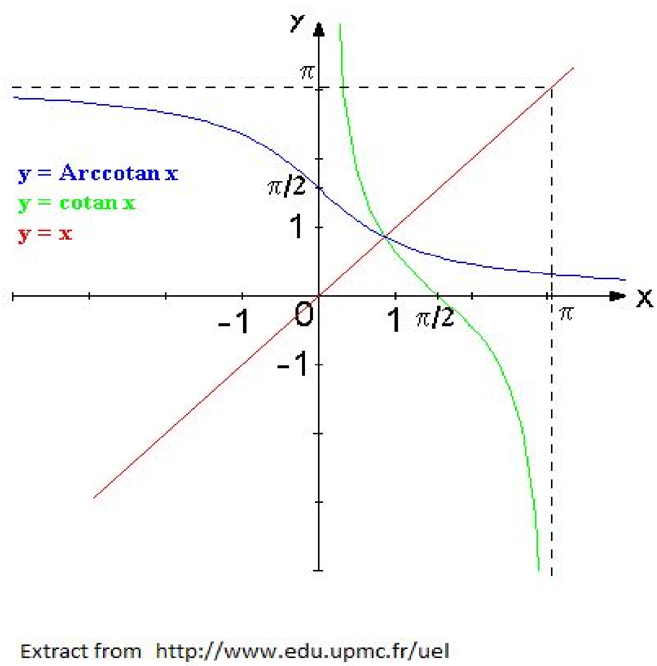

Figure 3.19

We list some of the basic properties of the arccotangent application.
Properties
$arccot(x) = y$ if and only if $y \in]0, \pi[$ and $\cot(y) = x$.
$\cot(arccot(x)) = x$ provided $x \in \mathbb{R}$.
$arccot(\cot(x)) = x$ provided $x \in]0, \pi[$.
Exercise: Find the exact values of the following.

1. $\arctan(\sqrt{3})$.

2. $arccot(-\sqrt{3})$.

3. $\cot(arccot(-5))$

Solution:

1. We know $\arctan(\sqrt{3})$ is the real number $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ with $\tan(y) = \sqrt{3}$. We find $y = \frac{\pi}{3}$, so $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

2. The real number $y = arccot(-\sqrt{3}) \in]0, \pi[$ with $\cot(y) = -\sqrt{3}$. We get $arccot(-\sqrt{3}) = \frac{5\pi}{7}$.

3. We can apply properties of the arccotangent application directly and obtain $\cot(arccot(-5)) = -5$.

Chapter 4

Real-Valued Functions of a Real Variable

In this chapter we shall study limit, continuity and differentiability of real valued functions defined on certain sets.

4.0.5 Overview

In mathematics, an "*elementary function*" is a function of a single variable composed of particular simple functions.

Basic examples:

The elementary functions of (x) of mathematics comprise:

- *Polynomial functions:* $x \mapsto \ a_0 + a_1x + a_2x^2 + ... + a_nx^n, a_i \in \mathbb{R}, i = 0, ..., n$
- *Trigonometric functions:* $x \mapsto \sin x,\ \cos x,\ \tan x,\ \cot x$
- *Exponential functions:* $x \mapsto \exp x$
- *Logarithms:* $x \mapsto \ln x$
- *Inverse trigonometric functions*: $x \mapsto \arcsin x,\ \arccos x, \arctan x, arccotx$
- *Hyperbolic functions* :

  $$x \mapsto \ \cosh x = \frac{\exp{(x)} + \exp{(-x)}}{2}, \quad x \mapsto \sinh x = \frac{\exp{(x)} - \exp{(-x)}}{2}$$
- *Inverse hyperbolic functions:* $x \mapsto \arg chx,\ \ x \mapsto \arg shx$.

4.1 Limit of a Function

The notion of a limit is a fundamental concept of calculus. More particularly, limits allow us to look at what happens in a **very**, **very** small region around a point.

Example 1: Values of $f(x) = \frac{x^2-4}{x-2}$ may be computed near $x = 2$

x	1.9	1.99	1.999 $\rightarrow$	$\leftarrow$ 2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999 $\rightarrow$	$\leftarrow$ 4.01	4.01	4.1

$$\lim_{x\to 2} f(x) = 4$$

Definition (Neighbourhood)

For $x_0 \in \mathbb{R}$, an open interval of the form $]x_0 - \delta, x_0 + \delta[$ for some $\delta > 0$ is called a **neighbourhood** of x_0.

4.1.1 Limit of a function at a point

Definition

A real valued function $f : D \to \mathbb{R}$ has "*limit value L as x tends to a finite value x_0*" if one can demonstrate that for any positive number ε (no matter how small), all the values $f(x)$ of the function will eventually be this close to the value L by restricting x to values very close, but not equal, to x_0. That is, one can produce a positive number δ so that if x, different from x_0, lies between $x_0 - \delta$ and $x_0 + \delta$ so then we can be sure that the value $f(x)$ lies between $L - \varepsilon$ and $L + \varepsilon$. Formally:

$$\forall \varepsilon > 0, \exists\, \delta > 0, \forall x \in D : x \in\,]x_0 - \delta, x_0 + \delta[\; \Rightarrow\; f(x) \in\,]L - \varepsilon, L + \varepsilon[$$

or we can write

$$\forall \varepsilon > 0, \exists\, \delta > 0, \forall x \in D : |x - x_0| < \delta \Rightarrow |\, f(x) - L| < \varepsilon$$

If a function $f(x)$ has a limit value L as x approaches a finite value x_0, we write:

$$\lim_{x\to x_0} f(x) = L$$

Example 1: Show that $\lim_{x\to 4} (2x - 1) = 7$. We have $f(x) = 2x - 1$, $x_0 = 4$ and $L = 7$ and the question we must answer is "how close should x be to 4 if want to be sure that $f(x) = 2x - 1$ differs less than ε from $L = 7$?"

To figure this out we try to get an idea of how big $|\, f(x) - L|$ is:

$| f(x) - L| \; = \; |(2x-1)-7| \; = 2\cdot|x-8|$
So, if $2\cdot|x-x_0|<\varepsilon$ then we have $| f(x)-L|<\varepsilon$, i.e.
$|x-x_0|<\frac{\varepsilon}{2} \implies |f(x)-L|<\varepsilon$.
We can therefore choose $\delta=\frac{\varepsilon}{2}$. No matter what $\varepsilon>0$ we are given our δ will also be positive,

$$\forall\varepsilon>0, \exists\ \delta>0\ , \forall x\in D\ :\ |x-4|<\delta \implies |(2x-1)-7|<\varepsilon$$

That shows that $\lim\limits_{x\to 4} f(x)=7$.

Definitions (Left limit and right limit)

(i) We say that f has the *left limit* $L\in\mathbb{R}$ as x tends to x_0 iff

$$\forall\varepsilon>0, \exists\ \delta>0, \forall x\in D: x\in\,]x_0-\delta,x_0[\ \Rightarrow\ f(x)\in\,]L-\varepsilon,L+\varepsilon[$$

and in that case we write:

$$\lim_{\substack{x\to x_0\\ <}} f(x)=L \quad \text{or} \quad \lim_{x\to x_0^-} f(x)=L$$

(ii) We say that f has the *right limit* $L\in\mathbb{R}$ as x tends to x_0 iff

$$\forall\varepsilon>0, \exists\ \delta>0, \forall x\in D: x\in\,]x_0,x_0+\delta[\ \Rightarrow\ f(x)\in\,]L-\varepsilon,L+\varepsilon[$$

and in that case we write:

$$\lim_{\substack{x\to x_0\\ >}} f(x)=L \quad \text{or} \quad \lim_{x\to x_0^+} f(x)=L$$

Theorem (Existence of the limit)

Let f be a real valued function defined on a set $D\subset\mathbb{R}$, then $\lim\limits_{x\to x_0} f(x)$ exists if and only if:

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0} f(x)$$

Example 1:
Let $f:[-1,1]\to\mathbb{R}$ defined by:

$$f(x)=\begin{cases} 0, & -1\le x\le 0\\ 1, & 0<x\le 1\end{cases}$$

$\lim_{x \to x_0} f(x)$ does not exist because $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 1$.

Example 2:

Let $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \sin\left(\frac{1}{x}\right)$$

We see in Figure 4.1, that $\sin\left(\frac{1}{x}\right)$ oscillates between $+1$ and -1 as $x \to 0$. This means that $f(x)$ gets close to any number between $+1$ and -1 as $x \to 0$, but that the function $f(x)$ never stays close to any particular value because it keeps oscillating up and down.

Here again, the limit $\lim_{x \to 0} f(x)$ does not exist.

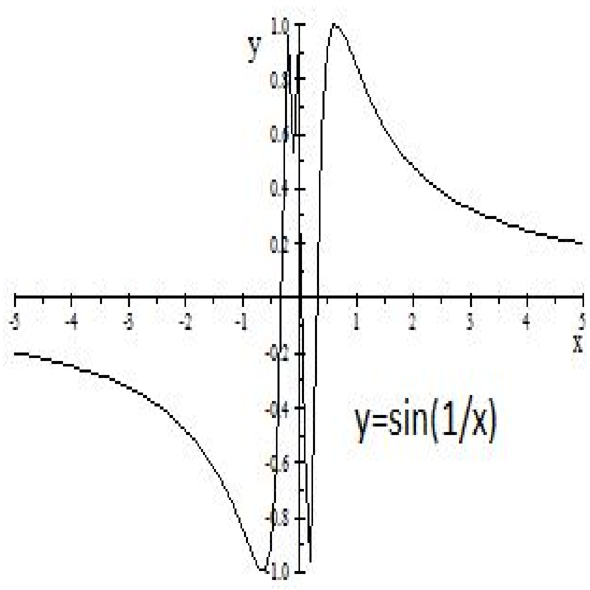

Figure 4.1

4.1.2 Limit of a function at infinity

Definition

A function f has limit value L as x becomes large if one can demonstrate that for any positive number ε there exists a positive number A, such that all the values $f(x)$ of the function lies between $L - \varepsilon$ and $L + \varepsilon$ for $x > A$.

We write: $\lim_{x\to+\infty} f(x) = L$. One can similarly define the notion of a limit as x becomes large and negative: $\lim_{x\to-\infty} f(x) = L$.

Example 1: Let's compute

$$\lim_{x\to\infty} \frac{5x^2+1}{4x^2+3x-2}$$

We divide the numerator and denominator by x^2, and you get

$$\lim_{x\to\infty} \frac{5x^2+1}{4x^2+3x-2} = \lim_{x\to\infty} \frac{5+\frac{1}{x^2}}{4+\frac{3}{x}-\frac{2}{x^2}} = \frac{5}{4}.$$

Example 2: Compute

$$\lim_{x\to\infty} \frac{x}{x^5-2}$$

We divide numerator and denominator by x^5. This leads to

$$\lim_{x\to\infty} \frac{\frac{1}{x^4}}{1-\frac{2}{x^5}} = \frac{0}{1} = 0.$$

4.1.3 Properties of the limit

The following properties remain true if one replaces each limit by a one-sided limit, or a limit for $x \to \infty$.

Let f and g be two given functions whose limits for $x \to x_0$ we know,

$$\lim_{x\to x_0} f(x) = L_1, \lim_{x\to x_0} g(x) = L_2.$$

Then:

1. $\lim_{x\to x_0} (f+g)(x) = L_1 + L_2$.
2. $\lim_{x\to x_0} (f \cdot g)(x) = L_1 \cdot L_2$.
3. $\lim_{x\to x_0} (\lambda \cdot f)(x) = \lambda \cdot L_1$.
4. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$, if $\lim_{x\to x_0} g(x) \neq 0$.

Theorem. Suppose that

$$f(x) \leq g(x) \leq h(x)$$

(for all x) and that

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x)$$

Then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x)$$

Corollary.

If $\lim_{x \to x_0} f(x) = 0$ and g is a bounded function. Then

$$\lim_{x \to x_0} (f \cdot g)(x) = 0$$

Example 3: Compute

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Since we have $-1 \quad \leq \sin\left(\frac{1}{x}\right) \leq \quad +1$ .Then

$$-x^2 \quad \leq x^2 \cdot \sin\left(\frac{1}{x}\right) \leq \quad +x^2,$$

Since

$$\lim_{x \to 0} - x^2 = \lim_{x \to 0} + x^2 = 0$$

The corollary tells us that

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

4.1.4 Indeterminate Forms

Definition. A function f is said to have an indeterminate form at x_0(where x_0 can be finite or infinite) if:

1. f is continuous on an interval including x_0, except possibly at x_0.
2. When we try to evaluate f at x_0 we obtain one of the following forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 1^{\infty}, \infty^{0}, 0^{\infty}$$

Here 0 and 1 represent **variable** quantities approaching the respective value, **NOT** constants with that value. Some indeterminate forms can be solved by rewriting the limit in an equivalent form by factoring through elimination, multiplying by the conjugate, by the trigonometric identities or using L'Hôpital's rule (See the next chapter).

Example 1: By foctoring

$$\begin{aligned}
\lim_{x\to -1}\frac{x^2-x-2}{x^2-2x-3} &= \frac{0}{0}\\
\lim_{x\to -1}\frac{x^2-x-2}{x^2-2x-3} &= \lim_{x\to -1}\frac{(x-2)(x+1)}{(x-3)(x+1)}\\
&= \lim_{x\to -1}\frac{(x-2)}{(x-3)}\\
&= \frac{3}{4}
\end{aligned}$$

Example 2: By the conjugate

$$\begin{aligned}
\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} &= \frac{0}{0}\\
\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4}\frac{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}{(x-4)\left(\sqrt{x}+2\right)}\\
&= \lim_{x\to 4}\frac{(x-4)}{(x-4)\left(\sqrt{x}+2\right)}\\
&= \lim_{x\to 4}\frac{1}{\left(\sqrt{x}+2\right)}\\
&= \frac{1}{4}
\end{aligned}$$

Example 3: By trigonometric identities

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(x)}{\sin(2x)} &= \frac{0}{0}\\
\lim_{x\to 0}\frac{\sin(x)}{\sin(2x)} &= \lim_{x\to 0}\frac{\sin(x)}{2\sin(x)\cos(x)}\\
&= \lim_{x\to 0}\frac{1}{2\cos(x)}\\
&= \frac{1}{4}
\end{aligned}$$

4.2 Continuous Functions

4.2.1 Continuity of a function at a point

Definitions.

- Let f be a real function on a subset of the real numbers. Then f is **continuous** at x_0 if

$$\begin{cases} 1) & x_0 \in D_f \\ 2) & \lim_{x\to x_0} f(x) = f(x_0) \end{cases}$$

- In particular, if the *left hand limit*, *right hand limit* and the value of the function at $x = x_0$ exist and are equal to each other, i.e.,

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

then f is said to be **continuous** at $x = x_0$.

- A function is **continuous** if it is continuous at every x_0 in its domain D_f.

- If it is not continuous there, i.e. if either the limit does not exist or is not equal to $f(x_0)$ we will say that the function is **discontinuous** at x_0.

Example 1: Consider the function

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \end{cases}$$

This function is continuous at all x_0, $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} |x| = |x_0| = f(x_0)$.
Example 2:
Let $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Then f is continuous at $x = 0$, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{\sin x}{x} = f(0) = 1$.

4.2.2 Continuity of a function in an interval

Definition.

1. f is said to be continuous in an open interval $]a, b[$ if it is continuous at every point x_0 in this interval.

2. f is said to be continuous in the closed interval $[a, b]$ if

    - f is continuous in $]a, b[$.
    - f is right continuous at a point a , i.e. $\lim\limits_{x \to a^+} f(x) = f(a)$.
    - f is left continuous at a point b , i.e. $\lim\limits_{x \to b^-} f(x) = f(b)$.

Property. All polynomials, rational functions, trigonometric functions, the inverse trigonometric functions, the absolute value function, the exponential and logarithm functions are continuous everywhere within its domain.

Example : The function $y = \frac{1}{x^2}$ is continuous for $x > 1$ or $x < -1$ but is not continuous on the interval $-1 < x < 1$ (See Figure 4.2).

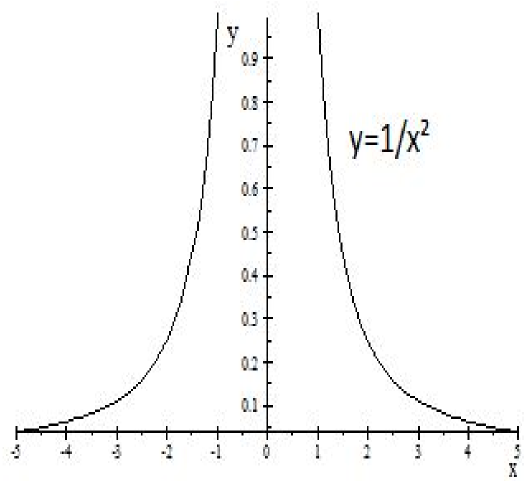

Figure 4.2

4.2.3 Continuous Extension at a point

We can redefine functions with removable discontinuities to obtain continuous functions.

Proposition.

Let I be an interval , and $x_0 \in I$. Let f be defined on $I - \{x_0\}$ such that $\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R}$.

Consider the function $\widetilde{f}$:

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{for} \quad x \in I - \{x_0\} \\ l & \text{for} \quad x = x_0 \end{cases}$$

then, the function $\widetilde{f}$ is a continuous at x_0.

Example 1: Find a continuous extension of the function $f(x) = \frac{\sin x}{x}$.

The domain of f is $D_f = \mathbb{R}^*$, then f is discontinuous at $x = 0$ because $f(0)$ is not defined. Since $\lim\limits_{x\to 0} f(x)$ exists, the discontinuity is removable.

We know that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. For the function to be continuous at zero we need to define $f(0)$ we make $f(0) = \lim\limits_{x\to 0} \widetilde{f}(x) = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

and redefine the function: $\widetilde{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{for} \quad x \neq 0. \\ 1 & \text{for} \quad x = 0. \end{cases}$

We say $\widetilde{f}$ is the continuous extension of f to $x = 0$.

4.2.4 The Intermediate Value Theorem

It is said that a function is continuous if you can draw its graph without taking your pencil o the paper.

A more precise version of this statement is the intermediate value theorem:

Theorem. *If a function f is continuous on a closed interval $[a, b]$, and if y_0 is some number between $f(a)$ and $f(b)$, then there is a number x_0 in the interval $[a, b]$ such that $f(x_0) = y_0$* (See Figure 4.3).

Example: Use the Intermediate Value Theorem to prove $x^2 = 2$ has a root.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by: $f(x) = x^2$.

The function f is continuous on a closed interval $[1, 2]$.

One has $f(1) = 2$ and $f(2) = 4$. Since $f(1) \leq 2 \leq f(2)$, the intermediate value theorem with $a = 1$, $b = 2$, $y_0 = 2$ tells us that there is a number x_0 between 1 and 2 such that $f(x_0) = 2$, i.e. for which $x_0^2 = 2$. So the theorem tells us that the *square root* of 2 exists.

4.2.5 Continuity of composite functions

Definition.

Let f and g be real valued functions such that $(f \circ g)$ is defined at x_0. If g is continuous at x_0 and f is continuous at $g(x_0)$, then $(f \circ g)$ is continuous at x_0.

Example: Since both $f(x) = x^2 + 1$ and $g(x) = \cos x$ are continuous on $\mathbb{R}$.

Therefore, both
$(f \circ g)(x) = \cos^2 x + 1$, and
$(g \circ f)(x) = \cos(x^2 + 1)$

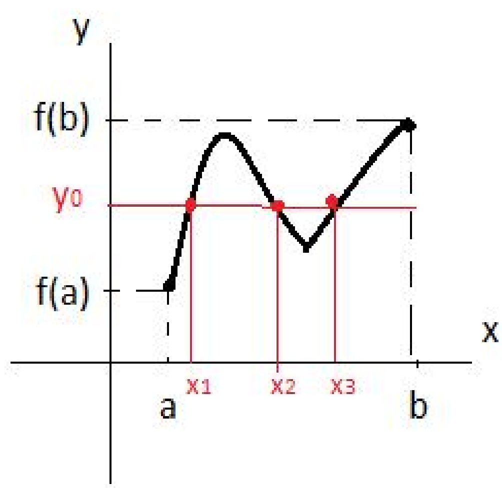

Figure 4.3

are continuous on $\mathbb{R}$.

4.2.6 Continuity of the algebraic combinations of functions

Definition.

If f and g are both continuous at x_0 and λ is any constant, then each of the following functions is also continuous at x_0: The sum $f + g$, the difference $f - g$, the constant multiple λf , the product $f \cdot g$, the quotient f/g , if $g(x_0) \neq 0$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Here is a continuous function on $\mathbb{R}$ because

- the inverse function of x , $x \mapsto \frac{1}{x}$ is a continuous on $\mathbb{R}^*$.
- the sine function $x \mapsto \sin x$ is a continuous on $\mathbb{R}$.

- the composite functions $x \mapsto \sin\left(\frac{1}{x}\right)$ is a continuous on $\mathbb{R}^*$
- the function $x \mapsto x$ is a continuous on $\mathbb{R}$.
- the product function $x \mapsto x\sin\left(\frac{1}{x}\right)$ is a continuous on $\mathbb{R}^*$.
- since $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = f(0)$, then the function f is a continuous at $x_0 = 0$.

4.3 Differentiability of Functions

4.3.1 Differentiability of a function at a point

Definition. **(Differentiability)**
Let f be a real valued function . Then f is said to be **differentiable** at $x_0 \in D_f$ if

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = l \in \mathbb{R}$$

in that case the value l is called the derivative of f at x_0.
The derivative of f at x_0, if exists, is denoted by $f^{'}(x_0)$.
Example 1: The function $f : x \longmapsto \sqrt{x}$ is differentiable at $x_0 = 1$.

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} &= \lim_{x\to 1} \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{(x-1)\left(\sqrt{x}+1\right)} \\ &= \lim_{x\to 1} \frac{x-1}{(x-1)\left(\sqrt{x}+1\right)} \\ &= \lim_{x\to 1} \frac{1}{\left(\sqrt{x}+1\right)} = \frac{1}{2} \end{aligned}$$

Since f is differentiable at $x_0 = 1$, then $f^{'}(1) = \frac{1}{2}$.
Example 2: The function $f : x \longmapsto \frac{1}{x}$ is differentiable at $x_0 = 2$.

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} &= \lim_{x\to 2} \frac{\frac{2-x}{2x}}{x-2} \\ &= \lim_{x\to 2} \frac{-1}{2x} = -\frac{1}{4} \end{aligned}$$

Since f is differentiable at $x_0 = 2$, then $f^{'}(2) = -\frac{1}{4}$.
Definitions (Left Differentiability and Right Differentiability)

- f is left differentiable at a point x_0 , i.e. $\lim\limits_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = f'_L(x_0)$.

- f is right differentiable at a point x_0 , i.e. $\lim\limits_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = f_R'(x_0)$.

Property. f is differentiable at a point x_0 iff f is left differentiable and right differentiable at this point, i.e. $f_L'(x_0) = f_R'(x_0)$.

Definition.

The function f is said to be **differentiable** at $x_0 \in D_f$ iff

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

Example 3: The derivative of $f(x) = x^2$ is $f'(x) = 2x$.

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} (2x+h) = 2x \end{aligned}$$

Theorem. (Differentiability implies continuity)

Suppose f is differentiable at $x_0 \in D_f$. Then f is continuous at x_0.

Proof. Note that

$$\begin{aligned} f \text{ is differentiable at } x_0 \in D_f \quad &\Rightarrow \quad \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) \\ &\Rightarrow \quad \lim_{h \to 0} f(x_0+h) - f(x_0) = \lim_{h \to 0} h \cdot f'(x_0) \\ &\Rightarrow \quad \lim_{h \to 0} f(x_0+h) - f(x_0) = 0 \\ &\Rightarrow \quad \lim_{h \to 0} f(x_0+h) = f(x_0) \\ &\Rightarrow \quad \lim_{x \to x_0} f(x) = f(x_0) \\ &\Rightarrow \quad f \text{ is continuous at } x_0 \end{aligned}$$

Remark. Every differentiable function is continuous, **but the converse is not true**.

Example 4: Consider the function

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This function is continuous at all x, but it is not differentiable at $x = 0$.

To see this try to compute the derivative at 0,

- $\lim\limits_{x \to 0^-} \frac{|x|-|0|}{x-0} = \lim\limits_{x \to 0^-} \frac{-x}{x} = -1 \Longrightarrow f_L'(0) = -1$.

- $\lim\limits_{x \to 0^+} \frac{|x|-|0|}{x-0} = \lim\limits_{x \to 0^+} \frac{x}{x} = 1 \Longrightarrow f_R'(0) = 1$.

- note that $f_L'(0) \neq f_R'(0)$.

4.3.2 Differentiability of a function in an interval

Definition.

1. f is said to be differentiable in an open interval $]a, b[$ if it is differentiable at every point x_0 in this interval.
2. f is said to be differentiable in the closed interval $[a, b]$ if
   - f is differentiable in $]a, b[$.
   - f is right differentiable at a point a , i.e. $\lim\limits_{x \to a^+} \frac{f(x)-f(x_0)}{x-x_0} = f^{'}(a)$.
   - f is left continuous at a point b , i.e. $\lim\limits_{x \to b^-} \frac{f(x)-f(x_0)}{x-x_0} = f^{'}(b)$.

4.3.3 Algebra of derivatives

If f, g are differentiable functions and λ is any constant, then

1. $(f+g)^{'}(x) = f^{'}(x) + g^{'}(x)$.
2. $(f \cdot g)^{'}(x) = f^{'}(x) \cdot g(x) + f(x) \cdot g^{'}(x)$
3. $(\lambda \cdot f)^{'}(x) = \lambda \cdot f^{'}(x)$.
4. $\left(\frac{f}{g}\right)^{'}(x) = \frac{f^{'}(x) \cdot g(x) + f(x) \cdot g^{'}(x)}{(g(x))^2}$, if $g(x) \neq 0$.

4.3.4 Derivatives of composite functions

Definition. If f and g are differentiable, so is the composition $f \circ g$. The derivative of $f \circ g$ is given by:

$$(f \circ g)^{'}(x) = f'(g(x)) \cdot g'(x).$$

Example: The function $f(x) = \sin 2x$ is the composition of two simpler functions, namely: $f(x) = g(h(x))$ where $g(u) = \sin u$ and $h(x) = 2x$. Since g and h are differentiable then $g^{'}(u) = \cos u$ and $h^{'}(x) = 2$.

Therefore the derivative of the composite functions rule implies that

$$f^{'}(x) = (g(h(x)))^{'} = g'(h(x)) \cdot h'(x) = (\cos 2x) \cdot 2 = 2\cos 2x.$$

4.3.5 Derivative of inverse function

Definition. If f is a function with inverse function f^{-1}, then

$$\left(f^{-1}\right)^{'}(y) = \frac{1}{f^{'}(x)}$$

Example: The inverse of the function $f(x) = x^2$ with reduced domain $[0, +\infty[$ is $f^{-1}(y) = \sqrt{y}$. Use the formula given above to find the derivative of f^{-1}. We have $f^{'}(x) = 2x$, so that $\left(f^{-1}\right)^{'}(y) = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$.

4.3.6 Table of Derivatives

$f(x)$	$f^{'}(x)$
a^x	$a^x \ln x$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$
$\cot x$	$\frac{-1}{\sin x} = -1 - \cot^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

$f(x)$	$f^{'}(x)$
c	0
x	1
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n.\sqrt[n]{x^{n-1}}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{x^n}$	$\frac{-n}{x^{n+1}}$
x^n	$n.x^{n-1}$
$\ln\|x\|$	$\frac{1}{x}$
$\exp x$	$\exp x$

4.3.7 Indeterminate Forms and L'Hospital's Rule

In this section, we will learn how to evaluate functions whose values cannot be found at certain points.

L'Hospital's Rule

Consider f and g are continuous functions on $[a, b]$ which are differentiable at every point in $]a, b[$, except possibly at $x_0 \in [a, b]$. Assume that:

1. $g(x) \neq 0$ and $g^{'}(x) \neq 0$ at every point in $[a, b]$.
2. $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$.
3. $\lim\limits_{x \to x_0} \frac{f^{'}(x)}{g^{'}(x)}$ exists

$$\text{Then} \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} \quad \text{exists} \quad \text{and} \quad \lim_{x \to x_0} \frac{f^{'}(x)}{g^{'}(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$$

Remark. Note that the rule is also valid for one-sided limits and for limits at infinity or negative infinity.

In fact, for the special case in which $f(x_0) = g(x_0) = 0$, $f^{'}$ and $g^{'}$ are continuous, and $g^{'}(x_0) \neq 0$, it is easy to see why the rule is true.

$$\begin{aligned}\lim_{x\to x_0} \frac{f^{'}(x)}{g^{'}(x)} &= \frac{f^{'}(x_0)}{g^{'}(x_0)} = \frac{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}} \\ &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} \\ &= \lim_{x\to x_0} \frac{f(x)}{g(x)}\end{aligned}$$

Example 1: Find $\lim_{x\to 1} \frac{\ln x}{x-1}$

$\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0}$

Thus, we can apply l'Hospital's Rule:

$$\begin{aligned}\lim_{x\to x_0} \frac{f^{'}(x)}{g^{'}(x)} = \lim_{x\to 1} \frac{(\ln x)^{'}}{(x-1)^{'}} &= \lim_{x\to 1} \frac{1}{x} \\ &= 1\end{aligned}$$

Then,

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{\ln x}{x-1} = 1$

Example 2: Find $\lim_{x\to +\infty} \frac{\exp x}{x^2}$

$\lim_{x\to +\infty} \frac{\exp x}{x^2} = \frac{\infty}{\infty}$

Thus, we can apply l'Hospital's Rule:

$\lim_{x\to x_0} \frac{f^{'}(x)}{g^{'}(x)} = \lim_{x\to +\infty} \frac{(\exp x)^{'}}{(x^2)^{'}} = \lim_{x\to +\infty} \frac{\exp x}{2.x} = \frac{\infty}{\infty}$

However, a second application of l'Hospital's Rule gives:

$\lim_{x\to x_0} \frac{f^{''}(x)}{g^{''}(x)} = \lim_{x\to +\infty} \frac{(\exp x)^{'}}{(2.x)^{'}} = \lim_{x\to +\infty} \frac{\exp x}{2} = +\infty$

Then,

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to +\infty} \frac{\exp x}{x^2} = \infty.$

Example 3: Find $\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x}$

$\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{0}$

If we blindly attempted to use l'Hospital's rule, we would get:

$\lim_{x\to x_0} \frac{f^{'}(x)}{g^{'}(x)} = \lim_{x\to \pi^-} \frac{(\sin x)^{'}}{(1-\cos x)^{'}} = \lim_{x\to \pi^-} \frac{\cos x}{\sin x} = -\infty$

Then,
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty.$$

4.3.8 Higher derivatives

If the derivative $f^{'}(x)$ of some function f exists for all x in the domain of f, then we have a new function, this function is called the **derivative function** of f, and it is denoted by $f^{'}$. Now that we have agreed that the derivative of a function is a function, we can repeat the process and try to differentiate the derivative. The result, if it exists, is called the **second derivative** of f. It is denoted $f^{''}$. The derivative of the second derivative is called the **third derivative**, written $f^{'''}$, and so on. The n-th derivative of f is denoted $f^{(n)}$. Thus

$$f^{(0)} = f, \quad f^{(1)} = f^{'}, \quad f^{(2)} = \left(f^{'}\right)^{'}, \quad ..., \quad f^{(n)} = \left(f^{n-1}\right)^{'}.$$

Example 1: If $f(x) = x^2 - x + 1$ then

$$\begin{array}{ccc} f(x) & = & x^2 - 2x + 3 \\ f^{'}(x) & = & 2x - 2 \\ f^{''}(x) & = & 2 \\ f^{'''}(x) & = & 0 \\ . & = & . \\ . & = & . \\ f^{(n)}(x) & = & 0. \end{array}$$

Example 2: If $f(x) = \exp x$ then
$f^{(1)}(x) = \exp x, \quad f^{(2)}(x) = \exp x, ..., \quad f^{(n)} = \exp x.$

Example 3. If $f(x) = \sin x$ then

$$\begin{aligned} f^{'}(x) &= \cos x \\ f^{''}(x) &= -\sin x \\ f^{'''}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ f^{(5)}(x) &= \cos x \\ f^{(6)}(x) &= -\sin x \\ f^{(7)}(x) &= -\cos x \end{aligned}$$

It's easy to find that ,

$$\sin^{(n)} x = \sin\left(x + \frac{n\pi}{2}\right)$$

4.3.9 Approximating functions by Taylor polynomials

First degree Taylor polynomials

If we know the function value at some point $f(x_0)$ and the value of the derivative at the same point $f^{'}(x_0)$, we can use these to find the tangent line, and then use the tangent line to approximate $f(x)$ for other points x.

The tangent line approximation of f for x near x_0 is called **the first degree Taylor polynomial** of f and is:

$$f(x) \approx f(x_0) + f^{'}(x_0)\,(x - x_0)\,. \qquad (\blacklozenge)$$

The statement that a complicated function behaves like a simpler function f for x near x_0 can be made more precise by use of the "O" notation. For example, we can replace the weak statement ($\blacklozenge$) by the stronger version,

$$f(x) = f(x_0) + f^{'}(x_0)\,(x - x_0) + O(x - x_0)$$

This means that there exists a function $\varepsilon(x)$ such that:

$$\lim_{x \to x_0} \frac{(x - x_0).\varepsilon(x)}{(x - x_0)} = 0$$

we write then

$$(x - x_0).\varepsilon(x) = O(x - x_0)$$

Example 1: Consider the function $f(x) = \sin x$. We want the first degree Taylor polynomial of this function near the point $x_0 = \frac{\pi}{4}$ and $x_0 = 0$.

- Since $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ and $(\sin)^{'}(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, the approximation of f for x near $x_0 = \frac{\pi}{4}$ is given by:

$$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + O(x - \frac{\pi}{4}), \quad \lim_{x \to \frac{\pi}{4}} O(x - \frac{\pi}{4}) = 0$$

- Since $\sin(0) = 1$ and $(\sin)^{'}(0) = \cos 0 = 1$, the approximation of f for x near $x_0 = 0$ is given by:

$$\sin x = x + O(x), \quad \lim_{x \to 0} O(x) = 0$$

Example 2: Consider the function $f(x) = \sqrt{x}$. We want the first degree Taylor polynomial of this function near the point $x_0 = 1$ and an evaluation of $\sqrt{1.002}$

the approximation of f for x near $x_0 = 1$ is given by:

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) + O(x-1), \quad \lim_{x \to 0} O(x-1) = 0$$

then

$$\sqrt{1.002} = 1 + \frac{1}{2}(1.002-1) + O(0.002) = 1.001 + O(0.002), \quad \lim_{x \to 0} O(0.002) = 0.$$

Higher order Taylor polynomials

The approximations of the function f by **the Taylor polynomial of degree** n, denoted by $P_n(x-x_0)$ for x near x_0 using more derivatives $f^{'}(x_0), f^{''}(x_0), ..., f^{(n)}(x_0)$ is given by: $f(x) = P_n(x-x_0) + R_n(x-x_0)$ for $\lim_{x \to x_0} R_n(x-x_0) = 0$.

Note that $R_n(x-x_0)$ is called **remainder term** which is the **approximation error** when approximating f with its Taylor polynomial. Using the "O" notation, the statement in Taylor polynomial reads as

$$R_n(x-x_0) = O\left((x-x_0)^n\right)$$

$$\boxed{f(x) = f(x_0) + \frac{f^{'}(x_0)}{1!}(x-x_0) + \frac{f^{''}(x_0)}{2!}(x-x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + O\left((x-x_0)^n\right)}$$

Example 1:

Consider the function $f(x) = \ln x$. We want a polynomial approximation of this function near the point $x_0 = 1$. The first few derivatives of f are

$$\begin{aligned} f(x) &= \ln x \\ f^{'}(x) &= \tfrac{1}{x} \\ f^{''}(x) &= \tfrac{-1}{x^2} \\ f^{'''}(x) &= \tfrac{2}{x^3} \\ f^{(4)}(x) &= \tfrac{-3}{x^4} \end{aligned}$$

The derivatives evaluated at $x_0 = 1$ are

$$f(0) = 0, \qquad f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

By Taylor's polynomial we have,

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ... + (-1)^{n-1}\frac{(x-1)^n}{n} + O\left((x-1)^n\right)$$

4.3.10 Mac-Laurin Polynomials

The approximations of f by the Taylor polynomial of degree n for x near $x_0 = 0$ is called the approximations of f by **Mac-Laurin polynomial** and is given by:

$$\boxed{f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + ... + \frac{f^{(n)}(0)}{n!}.x^n + O\left(x^n\right), \ \lim_{x\to 0} O\left(x^n\right) = 0.}$$

Example: Consider the function $f(x) = \cos x$. We want Mac-Laurin polynomial of this function near the point $x_0 = 0$. The first few derivatives of f are

$$\begin{aligned} f^{'}(x) &= \cos x \\ f^{''}(x) &= -\sin x \\ f^{'''}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ f^{(5)}(x) &= \cos x \\ f^{(6)}(x) &= -\sin x \end{aligned}$$

It's easy to find that ,

$$\cos^n(x) = \cos\left(x + \frac{n\pi}{2}\right)$$

Since $\cos(0) = 1$ and $\sin(0) = 0$ the Maclaurin polynomials of the cosine is,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^{2n}\frac{x^{2n}}{2n!} + O\left(x^{2n+1}\right)$$

The polynomial approximation of degree 4 is given by:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O\left(x^4\right)$$

The polynomial approximation of degree 5 is given by:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O\left(x^5\right)$$

The first statement informs us that there are terms of order x^4 in the expansion. The second statement is stronger as it informs us that there are no terms of order x^5.

Basic Mac-Laurin Polynomial

$\exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + O\left(x^n\right).$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{(-1)^n x^{2n+1}}{2n+1!} + O\left(x^{2n+2}\right).$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{(-1)^n x^{2n}}{2n!} + O\left(x^{2n+1}\right).$

$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + ... + \frac{\alpha(\alpha-1)....(\alpha-(n-1))x^n}{n!} + O\left(x^n\right).$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \frac{(-1)^n x^{n+1}}{n} + O\left(x^{n+1}\right).$

$\arccos x = \frac{\pi}{2} - x - \frac{1}{2}.\frac{x^3}{3.} - \frac{1.3}{2.4}.\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} - ... - \frac{1.3.5...(2n-1).}{2.4.6....2n}.\frac{x^{(2n+1)}}{(2n+1)} + O\left(x^{2n+2}\right).$

$\arcsin x = x + \frac{1}{2}.\frac{x^3}{3.} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + ... + \frac{1.3.5...(2n-1).}{2.4.6....2n}.\frac{x^{(2n+1)}}{(2n+1)} + O\left(x^{2n+2}\right).$

$\arctan g = x - \frac{x^3}{3.} - \frac{x^5}{5} - \frac{x^7}{7} + ... - \frac{x^{(2n+1)}}{(2n+1)} + O\left(x^{2n+2}\right).$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{2n}}{2n!} + O\left(x^{2n+1}\right).$

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{x^{2n+1}}{2n+1!} + O\left(x^{2n+2}\right)$

Chapter 5

Finite Expansions

5.1 Finite expansions at zero

Definition.

Let f be a real valued function. We said that the function f is represented by **a finite expansion at zero** if there exist real numbers $a_{0,}a_{1,}...,a_n$ and a real valued function ε such that

$$\boxed{f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + x^n\varepsilon(x),\ \lim_{x\to 0}\varepsilon(x) = 0.}$$

Then the function f is represented by the polynomial approximation of degree n, denoted by $P_n(x)$ for x near zero, which is called *the main part* of finite expansions at zero, such that: $P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$.

Remark: Note that $x^n\varepsilon(x) = O(x^n)$.

Example. Using the euclidean division by increasing power order, one has the finite expansion at zero of $f(x) = \frac{1}{1-x}$:

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + \frac{x^{n+1}}{1-x} = 1 + x + x^2 + ... + x^n + x^n.\left(\frac{x}{1-x}\right).$$

in this case $\varepsilon(x) = \frac{x}{1-x}$. We generally do not try to determine the function $\varepsilon(x)$.

Properties.

1. If the function f can be expanded at zero, then this expansion is *unique*.

2. If the function f can be expanded at zero, then $\lim_{x \to 0} f(x)$ exists and equal to a_0. This criterion is generally used to demonstrate that a function does not admit an expansion.

Example. The function $f(x) = \ln x$ does not have an expansion at zero, because $\lim_{x \to 0^+} f(x) = -\infty$.

5.2 Algebraic combinations of finite expansions

Definition.

If f and g can both be expanded at zero and λ is any constant, then each of the following functions is also can be expanded at zero: The sum $f + g$, the difference $f - g$, the constant multiple λf , the product $f \cdot g$, the quotient f/g , if $g(x_0) \neq 0$.

Consider the finite expansions at zero of f and g:

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + O(x^n)$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + ... + b_n x^n + O(x^n)$$

★ The finite expansion at zero of the sum $f + g$ is:

$$\begin{aligned} (f+g)(x) &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + ... + (a_n + b_n)x^n + O(x^n) \\ \lim_{x \to 0} \varepsilon(x) &= 0. \end{aligned}$$

★ The finite expansion at zero of the $f \cdot g$ is obtained by the product and keeping only the monomials of degree less than n in the product

$$\left(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n\right)\left(b_0 + b_1 x + b_2 x^2 + ... + b_n x^n\right)$$

★ The finite expansion at zero of the quotient f/g is obtained by the euclidean division of $\left(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n\right)$ by $\left(b_0 + b_1 x + b_2 x^2 + ... + b_n x^n\right)$ by increasing power order.

Example 1: Find the finite expansion at zero of $f(x) = \sinh x$ of the degree 4.

$$\begin{aligned}
\sinh x &= \frac{\exp x - \exp(-x)}{2} \\
&= \frac{1}{2}\left[\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) - \left(1 + \frac{-x}{1!} + \frac{x^2}{2!} + \frac{-x^3}{3!} + \frac{x^4}{4!}\right)\right] + O\left(x^4\right) \\
&= \frac{1}{2}\left(2x + 2\frac{x^3}{3!}\right) + O\left(x^4\right) \\
&= x + \frac{x^3}{3!} + O\left(x^4\right).
\end{aligned}$$

Example 2: Find the finite expansion at zero of $f(x) = \cos x \cdot \sin x$ of the degree 5.

We have $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O\left(x^5\right)$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O\left(x^5\right)$.

$$\begin{aligned}
f(x) &= \cos x \cdot \sin x \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) + O\left(x^5\right). \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + O\left(x^5\right). \\
&= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 + O\left(x^5\right).
\end{aligned}$$

Example 3: Find the finite expansion at zero of $f(x) = \frac{\sin x}{\cos x}$ of the degree 3.

Note that $\lim_{x \to 0} \cos x \neq 0$ then the quotient $f(x) = \frac{\sin x}{\cos x}$ can be expanded at zero.

Let $\sin x = x - \frac{x^3}{3!} + O\left(x^3\right)$ and $\cos x = 1 - \frac{x^2}{2!} + O\left(x^3\right)$.

Using the euclidean division by increasing power order we obtain:

$$\begin{aligned}
f(x) &= \frac{\sin x}{\cos x} \\
&= \frac{x - \frac{x^3}{3!}}{1 - \frac{x^2}{2!}} + O\left(x^3\right) \\
&= x + \frac{1}{3}x^3 + O\left(x^3\right).
\end{aligned}$$

Example 4: Find the finite expansion at zero of $f(x) = \frac{\ln(1+x)}{\sin x}$ of the degree 3.

Since $\sin x = x - \frac{x^3}{3!} + O\left(x^3\right)$ we have $\lim_{x \to 0} \sin x = 0$.

Note that the function f can be expanded at zero of the degree 3 if the finite expansions at zero of $\ln(1+x)$ and $\sin x$ are given of the degree 4.

$$\begin{aligned} f(x) &= \frac{\ln(1+x)}{\sin x} \\ &= \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O\left(x^4\right)}{x - \frac{x^3}{3!} + O\left(x^4\right)} \\ &= \frac{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + O\left(x^3\right)}{1 - \frac{x^2}{3!} + O\left(x^3\right)} \end{aligned}$$

Since $\lim_{x\to 0}\left(1 - \frac{x^2}{3!} + O\left(x^3\right)\right) = 0$ the function f can be expanded in this case.

$$\begin{aligned} \frac{\ln(1+x)}{\sin x} &= \frac{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + O\left(x^3\right)}{1 - \frac{x^2}{3!} + O\left(x^3\right)} \\ &= 1 - \frac{x}{2} + \frac{x^2}{6} - \frac{x^3}{12} + O\left(x^3\right). \end{aligned}$$

5.3 Composite of finite expansions

Definition.

If g can be expanded at zero of degree n and if f can be expanded at $g(0)$ of degree n such that $g(0) = 0$. Then the composite function $(f \circ g)$ can be expanded at zero of degree n by replacing the finite expansion of g in the finite expansion of f and by keeping only the monomials of degree $\leq n$.

Example 1: Find the finite expansion at zero of $f(x) = \exp(\cos x)$ of the degree 3.

If $g(x) = \cos x$ note that $g(0) \neq 1$.

We know that $\exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + O\left(x^3\right)$ and $\cos x = 1 - \frac{x^2}{2!} + O\left(x^3\right)$.

So if $g(x) = \cos x - 1 = -\frac{x^2}{2!} + O\left(x^3\right)$ in this case $g(0) = 0$.

$$\begin{aligned}
f(x) &= \exp(\cos x) \\
&= \exp(1 - 1 + \cos x) \\
&= \exp 1 \cdot \exp(-1 + \cos x) \\
&= \exp 1 \cdot \exp\left[-\frac{x^2}{2!} + O\left(x^3\right)\right] \\
&= \exp 1. \left[1 + \frac{\left(-\frac{x^2}{2!}\right)}{1!} + O\left(x^3\right)\right] \\
&= \exp 1 - \frac{\exp 1}{2} x^2 + O\left(x^3\right).
\end{aligned}$$

Example 2: Prove that the finite expansion at zero of $f(x) = \exp(\sin x)$ is given by $f(x) = \exp(\sin x) = 1 + x + \frac{x^2}{2} + O\left(x^3\right)$.

5.4 Finite expansions at a point

We said that the function $f : x \longmapsto f(x)$ can be represented by a finite expension at point x_0 if the function $F : X \longmapsto F(X)$ can be represented by finite expension at zero $X_0 = 0$ such that $F(X) = f(x_0 + X)$ and

$$F(X) = a_0 + a_1 X + a_2 X^2 + ... + a_n X^n + O(X^n), \lim_{X \to 0} O(X^n) = 0.$$

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ... + a_n(x - x_0)^n + O((x - x_0)^n),$$

and $\lim\limits_{x \to x_0} O((x - x_0)^n) = 0$.

Example 1: Find the finite expansion at a point $x_0 = 1$ of $f(x) = \exp x$ of the degree 3 .

$$\begin{aligned}
F(X) &= f(x_0 + X) \\
&= \exp(1 + X) \\
&= \exp 1 \cdot \exp X \\
&= \exp 1 \cdot \left[1 + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + O\left(X^3\right)\right] \\
&= \exp 1 \cdot \left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + O\left((x-1)^3\right)\right]
\end{aligned}$$

Example 2: Find the finite expansion at a point $x_0 = 2$ of $f(x) = \ln x$ of the degree 2 .

$$\begin{aligned} F(X) &= f(x_0 + X) \\ &= \ln(2 + X) \\ &= \ln\left[2 \cdot \left(1 + \frac{X}{2}\right)\right] \\ &= \ln 2 + \ln\left(1 + \frac{X}{2}\right) \\ &= \ln 2 + \frac{1}{2}X - \frac{1}{8}X^2 + O\left(X^2\right) \\ &= \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + O\left((x - 2)^2\right). \end{aligned}$$

5.5 Finite expansions at Infinity

We said that the function $f : x \longmapsto f(x)$ can be represented by a finite expension at infinity if the function $F : X \longmapsto F(X)$ can be represented by finite expension at zero $X_0 = 0$ such that $F(X) = f\left(\frac{1}{x}\right)$ and

$$F(X) = a_0 + a_1X + a_2X^2 + ... + a_nX^n + O(X^n), \lim_{X\to\infty} O(X^n) = 0.$$

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^n} + O\left(\frac{1}{x^n}\right).$$

Example 1: Find the finite expansion at infinity of $f(x) = \cos\frac{1}{x}$.
Let $X = \frac{1}{x}$ and thus:

$$\begin{aligned} \cos\frac{1}{x} &= \cos X = 1 - \frac{X^2}{2!} + \frac{X^4}{4!} + ... + \frac{(-1)^n X^{2n}}{2n!} + O\left(X^{2n}\right) \\ &= 1 - \frac{1}{2!x^2} + \frac{1}{4!x^4} + ... + \frac{(-1)^n}{2n!x^{2n}} + O\left(\frac{1}{x^{2n}}\right) \end{aligned}$$

5.6 Using finite expansions to evaluate limits

The finite expansions provide a good way to understand the behaviour of a function near a specified point and so are useful for solving some indeterminate forms. When taking a limit as $x \to 0$, we can often simplify the statement by substituting in finite expansions that we know.

Example 1: Find the limit $\lim\limits_{x\to 0}\frac{\exp(2x)\sin 3x}{\sinh(-2x)}$

$$\lim_{x\to 0}\frac{\exp(2x)\sin 3x}{\sinh(-2x)} = \lim_{x\to 0}\frac{\exp(2x)\cdot(3x+O(x))}{-2x+O(x)} = -\frac{2}{3}.$$

Example 2: Find the limit $\lim\limits_{x\to 0}\frac{1-\cos x}{\sin^2 x}$

$$\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x} = \lim_{x\to 0}\frac{\frac{x^2}{2}+O(x)}{(x+O(x))^2} = \frac{1}{2}.$$

Example 3: Find the limit $\lim\limits_{x\to 0}\frac{\exp x-1-x-\frac{x^2}{2}}{x^3}$

$$\exp x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+O\left(x^3\right)$$

We get $\exp x-1-x-\frac{x^2}{2} = \frac{x^3}{3!}+\frac{x^4}{4!}+O\left(x^4\right)$
and consequently

$$\begin{aligned}\frac{\exp x-1-x-\frac{x^2}{2}}{x^3} &= \frac{\frac{x^3}{3!}+\frac{x^4}{4!}+O\left(x^4\right)}{x^3}\\ &= \frac{1}{3!}+\frac{x}{4!}+O(x)\end{aligned}$$

so

$$\lim_{x\to 0}\frac{\exp x-1-x-\frac{x^2}{2}}{x^3} = \lim_{x\to 0}\frac{1}{3!}+\frac{x}{4!}+O(x) = \frac{1}{6}.$$

Chapter 6

Vector Space and Linear Maps

6.1 Vector Space

Underlying every **vector space** (to be defined shortly) is **a scalar field** K.

A **field** is a set of elements where the four basic operations $+, -, \times, \div$ are defined, with their usual properties (commutativity, associativity, distributivity). Examples of fields include the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. However, $\mathbb{N}$ is not a field (we cannot subtract or divide) and $\mathbb{Z}$ is not a field (we cannot divide).

Definition. (**Vector space)**

A **vector space** over a field K is a nonempty set V of objects, called vectors, on which are defined two operations:

1) *An internal operation* (*vector addition)*

$$\begin{array}{rccc} + : & V \times V & \longrightarrow & V \\ & (x, y) & \longmapsto & x + y \end{array}$$

2) *An external scalar* (*scalar multiplication)*

$$\begin{array}{rccc} \cdot : & K \times V & \longrightarrow & V \\ & (\alpha, x) & \longmapsto & \alpha x \end{array}$$

such that the following properties are satisfied:

1. $\forall u, v \in V$, $u + v = v + u$.

2. $\forall u, v, w \in V : (u + v) + w = u + (v + w)$.

3. $\forall u \in V, \exists\ 0_V \in V$: $u + 0 = u$ (0_V is the zero vector).
4. $\forall u \in V,\ \exists\ (-u) \in V\ :\ u + (-u) = 0_V$.
5. $\forall u, v \in V,\ \forall \alpha \in K\ :\ \alpha\,(u + v) = \alpha u + \alpha v$.
6. $\forall u \in V, \forall \alpha, \beta \in K : (\alpha + \beta)\, u = \alpha u + \beta u$.
7. $\forall u \in V, \forall \alpha, \beta \in K : (\alpha\beta)\, u = \alpha\,(\beta u)\,.$
8. $\forall u \in V : 1u = u$.

Examples:

1. $K^n = \{(x_1, x_2, .., x_n)\,/ x_i \in \mathbb{R}, i = 1, .., n\}$ is a vector space over the field K and K is any field (typically $K = \mathbb{R}$ or $K = \mathbb{C}$) with the vector addition and scalar multiplication defined as follows for all $(x_1, x_2, .., x_n)$ and $(y_1, y_2, .., y_n)$ from K^n and $\alpha \in K$:

$$\begin{aligned}(x_1, x_2, .., x_n) + (y_1, y_2, .., y_n) &= (x_{1+}y_1, x_{2+}y_2, .., x_n + y_n)\\ \alpha\,(x_1, x_2, .., x_n) &= (\alpha x_1, \alpha x_2, .., \alpha x_n)\end{aligned}$$

2. The set $P[x] = \left\{\underset{i=1}{\overset{n}{}} a_i x^i /\ a_i \in \mathbb{R}, i = 1, .., n\right\}$ of all polynomials over a field $\mathbb{R}$ is a vector space over $\mathbb{R}$ with the vector addition and scalar multiplication defined as follows for all $p(x) = \underset{i=1}{\overset{n}{}} a_i x^i$ and $q(x) = \underset{i=1}{\overset{n}{}} b_i x^i$ from $P[x]$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned}\forall p(x), q(x) &\in P[x] : p(x) + q(x) = \underset{i=1}{\overset{n}{}}\,(a_i + b_i)\, x^i\\ \forall p(x) &\in P[x], \forall \alpha \in \mathbb{R} : \alpha p(x) = \underset{i=1}{\overset{n}{}}\,(\alpha a_n)\, x^n\end{aligned}$$

3. The set V of all real valued continuous (differentiable or integrable) functions defined on the closed interval $[a, b]$ is a real vector space with the vector addition and scalar multiplication defined as follows:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

For all $f, g \in V$ and $\alpha \in \mathbb{R}$.

6.1.1 Subspaces of a vector space

Definition 1. (Subspace)

A subspace of a vector space V is a nonempty subset F of V that has three properties:

1. $0_V \in F$.
2. F is closed under vector addition. That is, $\forall u, v \in F : u + v \in F$.
3. F is closed under multiplication by scalars. That is, $\forall u \in F, \forall \alpha \in K$: $\alpha u \in F$.

Definition 2. (Subspace)

A subspace of a vector space V is a a nonempty subset F of V if and only if:

$$\forall u, v \in F, \forall \alpha, \beta \in K : \alpha u + \beta v \in F.$$

Remark 1.

Properties (1), (2), and (3) guarantee that a subspace F of V is itself a vector space, under the vector space operations already defined in V.

Example 1:

- $\mathbb{R}^{n-1}$ is a subspace of $\mathbb{R}^n$.
- $\{0_V\}$ is a subspace of V.
- V is a subspace of V.

Example 2: Show that $F = \{(0, y, z), y, z \in \mathbb{R}\}$ is a subspace of real vector space $\mathbb{R}^3$.

- $0_{\mathbb{R}^3} \in F$ then F is a nonempty subset of $\mathbb{R}^3$.
- Let $u = (0, y_1, z_1)$, $v = (0, y_2, z_2) \in F$ and $\alpha, \beta \in \mathbb{R}$. Then,

$$\begin{aligned} \alpha u + \beta v &= \alpha(0, y_1, z_1) + \beta(0, y_2, z_2) \\ &= (0, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \in F. \end{aligned}$$

Hence, F is a subspace of $\mathbb{R}^3$.

6.1.2 Linear combinations

Definition. (Linear combination)

Let V be a vector space. We say that the vector u is a **linear combination** of the vectors $v_1, v_2, .., v_n$ of V if

$$\exists\ \alpha_1, \alpha_2, .., \alpha_n \in K : u = \alpha_1 v_1 + \alpha_2 v_2 + .. + \alpha_n v_n.$$

Example: Express $u = (-2, 3)$ in $\mathbb{R}^2$ over $\mathbb{R}$ as a linear combination of the vectors $v_1 = (1, 1)$ and $v_2 = (1, 2)$.

Let α_1, α_2 be scalars such that

$$\begin{array}{cccc} & u & = & \alpha_1 v_1 + \alpha_2 v_2 \\ \Rightarrow & (-2, 3) & = & \alpha_1(1, 1) + \alpha_2(1, 2) \\ \Rightarrow & (-2, 3) & = & (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2) \\ \Rightarrow & \alpha_1 + \alpha_2 = -2 & \text{and} & \alpha_1 + 2\alpha_2 = 3 \\ \Rightarrow & \alpha_1 = -7 & \text{and} & \alpha_2 = 5 \end{array}$$

Hence, $u = -7v_1 + 5v_2$

6.1.3 Linear independence and linear dependence

Definition. (**Linear independence**)

Let the set $S = \{v_1, v_2, .., v_n\} \subset V$, a vector space. We say that S is **linearly independent** if all scalars $\alpha_1, \alpha_2, .., \alpha_n$ are zero for which $\alpha_1 v_1 + \alpha_2 v_2 + .. + \alpha_n v_n = 0_V$. That is

$$\forall \alpha_1, \alpha_2, .., \alpha_n \in K : \alpha_1 v_1 + \alpha_2 v_2 + .. + \alpha_n v_n = 0_V \Rightarrow \alpha_1 = \alpha_2 = .. = \alpha_n = 0_K.$$

Otherwise we say S is **linearly dependent.**

Definition. (**Linear dependence**)

The set $S = \{v_1, v_2, .., v_n\}$ is **linearly dependent** if there are scalars $\alpha_1, \alpha_2, .., \alpha_n$ not all zero for which

$$\alpha_1 v_1 + \alpha_2 v_2 + .. + \alpha_n v_n = 0_V$$

That is:

$$\exists\ \alpha_1, \alpha_2, .., \alpha_n \in K : \alpha_i \neq 0, i \in \{1, .., n\} \wedge \alpha_1 v_1 + \alpha_2 v_2 + .. + \alpha_n v_n = 0_V.$$

Example 1: The set $S = \{(-1, 0), (2, 1)\}$ is **linearly independent.**

Let $\alpha_1, \alpha_2 \in \mathbb{R}$:

$$\begin{aligned}\alpha_1(-1,0)+\alpha_2(2,1)=(0,0) &\Rightarrow (-\alpha_1+2\alpha_2,\alpha_2)=(0,0).\\ &\Rightarrow -\alpha_1+2\alpha_2=0 \text{ and } \alpha_2=0.\\ &\Rightarrow \alpha_1=\alpha_2=0.\end{aligned}$$

Example 2: The set $S=\{(1,0),(-2,0)\}$ is **linearly dependent.**
Let $\alpha_1,\alpha_2\in\mathbb{R}$:

$$\begin{aligned}\alpha_1(1,0)+\alpha_2(-2,0)=(0,0) &\Rightarrow (\alpha_1-2\alpha_2,0)=(0,0).\\ &\Rightarrow \alpha_1-2\alpha_2=0\\ &\Rightarrow \alpha_1=2\alpha_2.\\ \exists\alpha_1=1\wedge\alpha_2=\tfrac{1}{2} &\wedge\ \alpha_1(1,0)+\alpha_2(-2,0)=(0,0).\end{aligned}$$

Remark 2.

- $\{u\}$ is **linearly independent** $\Leftrightarrow u\neq 0_V$.
- $0_V\in S=\{v_1,v_2,..,v_n\}\Longrightarrow S$ is **linearly dependent.**

6.1.4 Generating sets

Definition. (Generating sets)
Given a vector space V, a finite set of vectors $S=\{v_1,v_2,..,v_n\}\subset V$ is called a system of **generators** if every vector $u\in V$ can be expressed as a linear combination of vectors of S :

$$\forall u\in V,\exists\ \alpha_1,\alpha_2,..,\alpha_n\in K: u=\alpha_1v_1+\alpha_2v_2+..+\alpha_nv_n.$$

and we write $V=[S]$.

Example 1: The set $S=\{(1,1,1),(2,2,0),(3,0,0)\}$ is a system of generators of $\mathbb{R}^3$.

Let $u=(x,y,z)$ be a vector, we check the scalars $\alpha,\beta,\gamma\in\mathbb{R}$ such that

$$u=(x,y,z)=\alpha(1,1,1)+\beta(2,2,0)+\gamma(3,0,0)$$

$$\Rightarrow\ \left\{\begin{array}{c} x=\alpha+2\beta+3\gamma\\ y=\alpha+2\beta\\ z=\alpha\end{array}\right.\Rightarrow\left\{\begin{array}{c}\alpha=z\\ \beta=\frac{y-z}{2}\\ \gamma=\frac{x-y}{3}\end{array}\right.$$

$$\Rightarrow\ u=z(1,1,1)+\frac{y-z}{2}(2,2,0)+\frac{x-y}{3}(3,0,0)$$

Example 2: Find generating set of the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$

$$(x,y)=x(1,0)+y(0,1)\Rightarrow\mathbb{R}^2=[\{(1,0),(0,1)\}]$$

6.1.5 Bases of a vector space

Definition. (Bases)

Given a vector space V, a finite set of vectors $S = \{v_1, v_2, .., v_n\} \subset V$ is called a **basis** of V if has two properties:

1. S is **linearly independent.**
2. S is a **generator of** V, that is $V = [S]$.

Examples: Let $\mathbb{R}^n$ be a vector space over $\mathbb{R}$.

- If $n = 1$, the basis of $\mathbb{R}$ is the set $S_1 = \{1\}$.
  - Since $\forall x \in \mathbb{R} : x = 1.x$ then $\mathbb{R}$ is generated by S_1.
  - $\forall \alpha \in \mathbb{R} : \alpha.1 = 0 \Rightarrow \alpha = 0$. Thus the set S_1 is linearly independent.
- If $n = 2$, the basis of $\mathbb{R}^2$ is the set $S_2 = \{(1, 0), (0, 1)\}$.
  - $\mathbb{R}^2$ is generated by S_2.
  - $\forall \alpha, \beta \in \mathbb{R} : \alpha. (1, 0) + \beta. (0, 1) = (\alpha, \beta) = (0, 0) \Rightarrow \alpha = \beta = 0$. Thus the set S_2 is linearly independent.
- If $n = 3$, the basis of $\mathbb{R}^3$ is the set $S_3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
  - Clearly S_3 is linearly independent and is a system of generators of $\mathbb{R}^3$.

Property. If S is a basis, every vector can be written as a linear combination of its elements in a *unique way.*

Example 1. Let the set $S = \{(1, 0), (1, 1)\}$, we can write any vector of $\mathbb{R}^2$ as a linear combination of $(1, 0)$ and $(1, 1)$ in a unique way.

Example 2. The set $S = \{(1, 0), (0, 1), (1, 1)\}$ is not a basis but it is a system of generators. In this case any of the three vectors can be removed because it can be expressed as a combination of the other two. The linear combinations are not unique:

$$(2, 3) = 1.(1, 0) + 2.(0, 1) + 1.(1, 1)$$

6.1.6 Dimension of a vector space

Definition. (**Dimension**)

If a vector space V has a basis with finite number of elements, then every other basis of V has the same number of elements. This number is called the dimension of V. We write

$$\dim V = n$$

Properties. if we know that a vector space has dimension n ($\dim V = n$), then:

- Every basis consists of exactly n vectors (but not every set of n vectors is a basis!).
- Every system of generators has to contain at least n vectors.
- If a system of generators consists of n vectors, then it is a basis.
- If a set of n vectors is linearly independent, then it is a basis.

Proposition. Let V be a vector space of dimension n, and let F is a subspace of a vector space V then $\dim F \leq \dim V$. Furthermore, if $\dim F = \dim V$ then $F = V$.

Example 1: $\mathbb{R}^n$ is a vector space of dimension n. A particular basis is the *canonical basis*:

$$\begin{aligned} e_1 &= (1, 0, 0, .., 0) \\ e_2 &= (0, 1, 0, .., 0) \\ &. \\ &. \\ e_n &= (0, 0, 0, .., 1) \end{aligned}$$

Example 2: The set of three vectors $\{(1, 2), (-1, 2), (3, 1)\}$ is not a basis of $\mathbb{R}^2$ because $3 > \dim \mathbb{R}^2 = 2$.

Example 3:

- $\dim \{0_V\} = 0$.
- Let $\mathbb{C}$ be a vector space over $\mathbb{R}$, then $\dim \mathbb{C} = 2$.
- Let P_n be a vector space of polynomial over $\mathbb{R}$, then $\dim(P_n) = n + 1$.

6.2 Linear Maps

Definition. (**Linear map**)

A **linear map** f from a vector space V into a vector space W is a rule that assigns to each vector x in V a unique vector $f(x)$ in W, such that:

$$\begin{array}{rccc} f: & V & \longrightarrow & W \\ & x & \longmapsto & f(x) \end{array}$$

1. $\forall x_1, x_2 \in V : f(x_1 + x_2) = f(x_1) + f(x_2)$.
2. $\forall x \in V, \forall \alpha \in \mathbb{R} : f(\alpha x) = \alpha f(x)$.

Example 1: The map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $f(x, y) = \left(x^2, x + y, 1\right)$ is not linear. We can easily find vectors for which the condition is false. For example:

$$\begin{array}{rcl} f\left((1,0)+(0,0)\right) & = & f\ (1,0) = (1,1,1) \\ f\ (1,0) + f\ (0,0) & = & (1,1,1) + (0,0,1) = (1,1,2) \end{array}$$

Hence, $f\left((1,0)+(0,0)\right) \neq f\ (1,0) + f\ (0,0)$

Example 2: $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $f(x, y) = (3x - y, 0, 2y)$ is linear map:

1. $f\left((x_1, y_1) + (x_2, y_2)\right) = f\ (x_1{+}x_2, y_1{+}y_1) = \left(3\left(x_1 + x_2\right), 0, 2\left(y_1 + y_2\right)\right) = (3x_1 - y_1, 0, 2y_1){+}(3x_2 - y_2, 0, 2y_2) \Rightarrow f\left((x_1, y_1) + (x_2, y_2)\right) = f(x_1, y_1){+} f\left(x_2, y_2\right).$
2. $f\left(\alpha.(x, y)\right) = f\left(\alpha.x, \alpha.y\right) = \left(3\left(\alpha x\right) - \left(\alpha y\right), 0, 2\left(\alpha.y\right)\right) = \alpha.\left(3x - y, 0, 2y\right) = \alpha f(x, y)$.

Properties. Here are some simple properties of linear maps $f : V \to W$.

1. $f(0_V) = 0_W$.
2. $f(-x) = -f(x)$.
3. If V_1 is a subspace of V, then $f(V_1)$ is a subspace of V.
4. If W_1 is a subspace of W, then $f^{-1}(W_1)$ is a subspace of W.
5. The composite map of two linear maps is a linear map.

6.2.1 Linear maps and dimension

The kernel of a linear map

Definition. (**Kernel**)

The **kernel** (or **null space**) of such a f , denoted by $\ker f$, is the set of all x in V such that $f(x) = 0_W$ (the zero vector in W):

$$\ker f = \{x \in V, f(x) = 0_W\} = f^{-1}(\{0_W\})$$

The image of a linear map

Definition. (**Image**)

The **image** of f , denoted by Imf, is the set of all vectors in W of the form $f(x)$ for some x in V.

$$Imf = \{f(x), x \in V\} = f(V)$$

Properties. Let $f : V \to W$ be a linear map.

1. The kernel of f is a subspace of V.
2. The image of f is a subspace of W.

Proposition. Let $f : V \to W$ be a linear map.

1. f is injective if and only if $\ker f = \{0_V\}$.
2. f is surjective if and only if $Imf = W$.

Example: The map $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined as $f(x, y, z) = (x + y, z)$ is not injective and surjective.

- f is injective $\Leftrightarrow \ker f = \{0_{\mathbb{R}^3}\}$.

$$\begin{aligned}
\ker f &= \left\{X \in \mathbb{R}^3, f(X) = (0,0)\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3, f(x,y,z) = (0,0)\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3, (x+y,z) = (0,0)\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3, x+y = 0 \text{ and } z = 0\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3, x = -y \text{ and } z = 0\right\} \\
&= \{(-y,y,0), y \in \mathbb{R}\}
\end{aligned}$$

For example: $(-1, 1, 0) \in \ker f \Rightarrow \ker f \neq \{0_{\mathbb{R}^3}\}$. Hence f is not injective.

- f is surjective $\Leftrightarrow Imf=\left\{f(x), x\in\mathbb{R}^3\right\}=\mathbb{R}^2$.

$$\begin{aligned} Imf &= \left\{f(X), X\in\mathbb{R}^3\right\} \\ &= \{(x+y,z) : x,y,z\in\mathbb{R}\} \\ &= \{x\,(1,0)+y\,(1,0)+z\,(0,1) : x,y,z\in\mathbb{R}\} \end{aligned}$$

Hence, Imf is generated by two vectors $(1,0)\,,(1,0)$ which are the *canonical basis* of $\mathbb{R}^2$. Then $Imf=\mathbb{R}^2$ and f is surjective.

Proposition. Let $f: V\to W$ be a linear map, with V finite-dimensional. Then:

$$\dim V=\dim\ker f+\dim Imf$$

The rank of a linear map

Definition. (Rank)

The rank of a linear map f is the dimension of its image, written rank$f$:

$$\text{rank}f=\dim Imf$$

Example: Find $\ker f$, Imf and rank$f$ of the map $f:\mathbb{R}^4\to\mathbb{R}^3$ defined as $f(x,y,z,t)=(x-y,z+t,x-y+z)\,.$

$$\begin{aligned} \ker f &= \left\{X\in\mathbb{R}^4, f(X)=(0,0,0)\right\} \\ &= \left\{(x,y,z,t)\in\mathbb{R}^4, f(x,y,z,t)=(0,0,0)\right\} \\ &= \left\{(x,y,z,t)\in\mathbb{R}^4, (x-y,z+t,x-y+z)=(0,0)\right\} \\ &= \left\{(x,y,z,t)\in\mathbb{R}^4, x-y=0\ \wedge\ z+t=0\wedge x-y+z=0\right\} \\ &= \left\{(x,y,z)\in\mathbb{R}^3,\quad x=y\ \wedge\ z=t=0\right\} \\ &= \{(x,x,0,0)\,,\quad x\in\mathbb{R}\}=\{x.\,(1,1,0,0)\,, x\in\mathbb{R}\ \ \} \end{aligned}$$

Hence , $\ker f=[\{(1,1,0,0)\}]$.

$$\begin{aligned} Imf &= \{f(x,y,z,t)/\ \ x,y,z,t\in\mathbb{R}\} \\ &= \{(x-y,z+t,x-y+z)\,/\ \ x,y,z,t\in\mathbb{R}\} \\ &= \{(x-y)\,.\,(1,0,1)+t.\,(0,1,0)+z\,(0,1,1)\,/\ \ x,y,z,t\in\mathbb{R}\} \end{aligned}$$

Hence , $Imf=[\{(1,0,1)\,,(0,1,0)\,,(0,1,1)\}]$

Let $\alpha_1,\alpha_2,\ \alpha_3\in\mathbb{R}$:

$$\begin{aligned} \alpha_1\,(1,0,1)+\alpha_2\,(0,1,0)+\alpha_3\,(0,1,1)=(0,0,0) &\Rightarrow\ (\alpha_1,\alpha_2+\alpha_3,\alpha_1+\alpha_3)=(0,0,0)\,. \\ &\Rightarrow\ \begin{cases}\alpha_1=0\ . \\ \alpha_2+\alpha_3=0. \\ \alpha_1+\alpha_3=0.\end{cases} \\ &\Rightarrow\ \alpha_1=\alpha_2=\alpha_3=0. \end{aligned}$$

Therefore, the set $\{(1,0,1),(0,1,0),(0,1,1)\}$ is linearly independent and it is the basis of Imf.

$$\text{rank} f = \dim Imf = 3.$$

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