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Pedagogic Handout

Courses in Mathematics 2

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FOR FIRST YEAR STUDENTS IN SCIENCE AND TECHNOLOGY

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Introduction

This Handout, the result of teaching experience, is aimed at first year students in Science and Technology. Its objective is to guide students efficiently through the fundamental principles and computation techniques to be qualified to deal with mathematical problems.

Five chapters are included in this Handout:

Chapter 1 deals with Matrices, we learn about matrices, we are studying operations on the matrices, types of matrices, inverse of a matrix, also, a second method (Gauss-Jordan elimination method) for finding a matrix inverse will be outlined. we will learn about a characteristic quantity associated with square matrices-the determinant.

The determinant which studied in Chapter 2 plays an important role in matrix calculus and solving linear systems. It allows us to know whether a matrix is invertible or not. We start by giving the expression for determinant of a matrix and also cofactor method of finding the inverse of a square matrix.

Systems of linear equations and their solutions constitute one of the major topics that we will study in the chapter 2. In the first section we will introduce some basic terminology then, We have a brief discussion of methods for solving such systems.

Chapter 3 deals with Integrals end primitive functions.

Integration started as a method to solve problems in mathematics and physics, such as finding the area under a curve, or determining displacement from velocity. Today integration is used in a wide variety of scientific fields.

Integrals refer to the concept of an anti-derivative, a function whose derivative is the given function; in this case, they are also called indefinite integrals.

Both the integral and differential are related to each other by the fundamental theorem of calculus, which provides a method to compute the definite integral of a function when its anti-derivative is known.

In this chapter, we will learn about some important methods for calculating integrals.

In the fourth chapter, we will study the ordinary differential equations and their corresponding methods of solution, especially first-order ordinary differential equations, as well as second-order differential equations with constant coefficients.

The last chapter is devoted to the functions of several variables. We will present some basic definitions of derivatives and multiple integrals, especially the double integral.

Finally, we hope that this cours will be a means to help students understand the lessons and be able to apply them.

Contents

Chapter 1 Matrices and Determinants

Introduction

- ❏ *Definition of a Matrix*
- ❏ *Special Matrices*
- ❏ *Operations on Matrices*
- ❏ *The inverse of a matrix*
- ❏ *Matrix of a linear application*
- ❏ *Linear map of matrix*
- ❏ *Change of basis (Transit matrix)*
- ❏ *Determinant of a Square Matrix*
  - *Properties of Determinants*
  - *Det of particular matrices*
  - *Inverse of a matrix using cofactor*
- ❏ *Rank of a Matrix*

1.1 Definition of a Matrix

Definition 1.1 (Matrix)

A matrix A is a rectangular array $(table)$ of elements of $\mathbb{k}$. See [3]

- *It is said to be of order $m \times n$ if the table has m rows and n columns, enclosed within a bracket $($either round or square$)$*
- *The numbers m and n are called the dimensions of the matrix.*
- *The numbers in the table are called the coefficients of A.*
- *The coefficient in the i^{th}row and in the j^{th} column is denoted by a_{ij}.*
- *The matrices is denoted with capital letters, like A, B, C etc.* ♣

A short hand method of writing a general $m \times n$ matrix is the following.

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n} = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} \tag{1.1}$$

Remark

- $\mathbb{k}$ denotes a field wich is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.
- If $m = n$, in equation (1.1) we say that matrix A is of order n.
- A matrix having only one column is called column matrix (or vector column), and a matrix with only one row is called a row matrix (or row vector).
- We denote the set of matrices of order $m \times n$ by $\mathcal{M}_{m\times n}(\mathbb{k})$.

Example 1.1

Let be the following matrices

$$A = \begin{pmatrix} 1 & 0 & 7 \\ 2 & -3 & 0 \end{pmatrix}, \ B = \begin{pmatrix} -5 & 0 \\ 7 & -3 \\ 3 & 4 \end{pmatrix}, \ C = \begin{pmatrix} 9 & -3 & 1 \end{pmatrix}, D = \begin{pmatrix} -8 \\ 2 \\ 0 \\ -5 \end{pmatrix}$$

A is a matrix of order 2×3, with two rows and three columns,

such as $a_{11} = 1, \quad a_{12} = 0, \quad a_{13} = 7, \quad a_{21} = 2, \quad a_{22} = -3, \quad a_{23} = 0.$

B is a matrix of order 3×2, $B \in \mathcal{M}_{3\times 2}(\mathbb{R})$.

C is a row matrix (row vector), with three columns.

D is a column matrix (column vector), with four rows.

1.1.1 Equality of two matrices

Definition 1.2

[3, 5] Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal if:

- *They possess the same number of rows and the same number of columns*
- *If $a_{ij} = b_{ij}$, $\forall i = 1, ..., m$ and $\forall j = 1, ..., n$.*

♣

Exercise 1.1

Set the coefficients α and β that the two matrices A and B are equal

$$A = \begin{pmatrix} -7 & \alpha \\ -9 & -\beta \end{pmatrix}, B = \begin{pmatrix} -7 & 2 \\ -9 & 5 \end{pmatrix}$$

Solution

$$\begin{pmatrix} -7 & \alpha \\ -9 & -\beta \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \Longleftrightarrow \begin{cases} \alpha = 2 \\ -\beta = 5 \Longrightarrow \beta = -5 \end{cases}$$

1.2 Special Matrices

1.2.1 Zero-matrix

Definition 1.3

[3, 6] If every coefficient in a matrix $A_{m\times n} = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$ in (1.1) is zero, it is known as a zero matrix and denoted by $0_{n\times m}$. ♣

For example:

$$0_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; 0_{3\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1.2.2 Square matrix

Definition 1.4

[3, 6] A matrix that has equal numbers of rows and columns $(n = m)$ is known as a square matrix. We call this matrix A of order n, and is represented by n only i.e. $A \in \mathcal{M}_n(\mathbb{k})$. ♣

For example

$$A = \begin{pmatrix} 2 & 1 \\ -\sqrt{3} & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}), \; B = \begin{pmatrix} 1 & 0 & 9 \\ 2 & -3 & \sqrt{2} \\ 5 & 4 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

1.2.3 Diagonal matrix

Definition 1.5

A square matrix $A = (a_{ij})$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$, and at least one element $a_{ii} \neq 0$. See [6, 20]. ♣

For example

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}.$$

1.2.4 Identity Matrix

Definition 1.6

[3, 5] A square matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n, in other words $a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$. ♣

For example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

1.2.5 Upper Triangular matrix

Definition 1.7

A square matrix $A = (a_{ij})$ is said to be an upper triangular matrix if $a_{ij} = 0$ for $i > j$ See [3, 6]. ♣

For example

$$A = \begin{pmatrix} 1 & 2 & -5 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

1.2.6 Lower Triangular matrix

Definition 1.8

A square matrix $A = (a_{ij})$ is said to be an lower triangular matrix if $a_{ij} = 0$ for $i < j$ See [3, 6]. ♣

For example

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 9 & -3 & 0 \\ 3 & \sqrt{13} & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

1.2.7 Symmetric Matrix

Definition 1.9

A square matrix $A = (a_{ij})$ said to be a symmetric if $a_{ij} = a_{ji}$ for all i and j See [3, 5]. ♣

For example

$$A = \begin{pmatrix} 2 & 5 & -7 \\ 5 & -3 & \sqrt{3} \\ -7 & \sqrt{3} & 1 \end{pmatrix} \text{ is a symmetric Matrix.}$$

1.3 Operations on Matrices

The matrix operations include the addition, subtraction, multiplication of matrices, transpose of a matrix, and inverse of a matrix.

1.3.1 Scalar multiple of a matrix

Definition 1.10 (Scalar multiple of a matrix)

Let $A = (a_{ij})\ \mathcal{M}_{m\times n}(\mathbb{R})$*, then for any scalar* $\lambda \in \mathbb{R}$*, we defind* λA *by:* $\lambda A = (\lambda a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$,

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}$$

See [3, 11] ♣

For example

$$A = \begin{pmatrix} 3 & 0 & -5 \\ 1 & -5 & \sqrt{2} \\ 5 & 2 & 3 \end{pmatrix} \Rightarrow 3A = \begin{pmatrix} 9 & 0 & -15 \\ 3 & -15 & 3\sqrt{2} \\ 15 & 6 & 9 \end{pmatrix}$$

Remark

Multiplying a scalar by a matrix is commutative, ie $\lambda A = A\lambda$

1.3.2 Addition of Matrices

Definition 1.11 (Addition of Matrices)

Let $A = (a_{ij})$ *and* $B = (b_{ij})$ *be are two matrices with same order* $m \times n$*, Then the sum* $A + B$ *is defined by:* $A + B = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} + (b_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} = (a_{ij} + b_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$*, see [11]*

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

♣

Example 1.2

$$\text{Let } A = \begin{pmatrix} 0 & 5 & -9 \\ \sqrt{3} & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 9 \\ 2 & -3 & 4 \end{pmatrix},$$

Find $3A$, $A+B$, $3A-2B$.

Solution

$$3A = \begin{pmatrix} 0 & 15 & -27 \\ 3\sqrt{3} & 0 & 6 \end{pmatrix}, A+B = \begin{pmatrix} 1 & 5 & 0 \\ 2+\sqrt{3} & -3 & 6 \end{pmatrix}$$

$$\begin{aligned} 3A - 2B &= \begin{pmatrix} 0 & 15 & -27 \\ 3\sqrt{3} & 0 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 0 & -18 \\ 4 & -6 & 8 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 15 & -45 \\ 3\sqrt{3}-4 & 6 & -2 \end{pmatrix} \end{aligned}$$

Property

Let A, B and C be matrices of order $m \times n$, and let $\lambda_1, \lambda_2 \in \mathbb{R}$. Then [3]

- $A + B = B + A$
- $(A+B) + C = A + (B + C)$
- $(\lambda_1 + \lambda_2) A = \lambda_1 A + \lambda_2 A$
- $(\lambda_1 \lambda_2) A = \lambda_1 (\lambda_2 A)$

1.3.3 Multiplication of Matrices

Definition 1.12 (Multiplication of Matrices)

Let $A = (a_{ij}) \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $B = (b_{jk}) \in \mathcal{M}_{n\times p}(\mathbb{R})$. The product AB is a matrix $C = (c_{ik})$ of order $m \times p$, defined by: [20, 27]

$$\begin{aligned} c_{ik} &= \sum_{l=1}^{n} a_{il} \times b_{lk}, \text{ where} \\ 1 &\leq i \leq m,\ 1 \leq k \leq p,\ m, n, p \in \mathbb{N} \end{aligned}$$

♣

Example 1.3

Obtain the product AB if $A = \begin{pmatrix} 7 & -1 \\ 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 2 & 0 \end{pmatrix}$

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 7 & -1 \\ 2 & -3 \end{pmatrix} \times \begin{pmatrix} 2 & 4 & 1 \\ 3 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{bmatrix} 7 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 7 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} & \begin{bmatrix} 7 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 11 & 26 & 7 \\ -5 & 2 & 2 \end{pmatrix} \end{aligned}$$

Remark

1. The multiplication of a matrix A by a matrix B is defined only when the number of columns of the first matrix A equals the number of rows of the second matrix B.
2. In general, the matrix multiplication is not commutative: $AB \neq BA$.

Example 1.4

$$\text{Let } A = \begin{pmatrix} -2 & 4 \\ 3 & 5 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$$

We have:

$$AB = \begin{pmatrix} -2 & 4 \\ 3 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 6 & -22 \\ 13 & -11 \end{pmatrix}$$
$$BA = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} \times \begin{pmatrix} -2 & 4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 19 \\ -16 & -12 \end{pmatrix}$$

Thus, $AB \neq BA$

Property

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid, See [3].

- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$

- $AI_n = I_n A = A$
- $A^n = \underbrace{A \times A ... \times A}_{n\,factors}, n \in \mathbb{N}^*$

  In general

- $(A+B)^2 \neq A^2 + 2AB + B^2$
- $(A-B)^2 \neq A^2 - 2AB + B^2$
- $(A+B)(A-B) \neq A^2 - B^2$

1.3.4 Transpose of a Matrix

Definition 1.13 (Transpose of a Matrix)

The transpose of matrix $A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$, written A^T, or A^t(is the matrix obtained by writing the rows of A in order as the columns of A^t and writing the columns of A as the rows of A^t, see [20]. ♣

Example 1.5

For example:

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 9 & 1 \\ 2 & -3 & \sqrt{2} & 6 \\ 5 & 4 & 1 & -4 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 1 & 2 & 5 \\ 0 & -3 & 4 \\ 9 & \sqrt{2} & 1 \\ 1 & 6 & -4 \end{pmatrix}$$

Remark

If $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, then $A^t \in \mathcal{M}_{m\times n}(\mathbb{R})$.

1.3.5 Properties of the matrices transpose of a matrix

Let A and B be matrices, and let $\lambda \in \mathbb{R}$. [3, 6] Then

- $(A+B)^t = A^t + B^t$
- $(A^t)^t = A$
- $(\lambda A)^t = \lambda A^t$, λ is a scalar.

- $(AB)^t = B^t A^t$
- If $A^t = A$, the matrix A is symmetric

✍ **Exercise 1.2**

Show that the matrix C is symmetric, where:

$$C = \begin{pmatrix} 1 & 2 & 5 \\ 2 & -3 & 4 \\ 5 & 4 & 8 \end{pmatrix}$$

Solution

Taking the transpose of C

$$C^t = \begin{pmatrix} 1 & 2 & 5 \\ 2 & -3 & 4 \\ 5 & 4 & 8 \end{pmatrix};$$

Clearly $C^t = C$ *so C is a symmetric matrix.*

1.3.6 Main diagonal

Definition 1.14 (Main diagonal)

The Main diagonal (Principal diagonal) of a square matrix, $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$, *is the list of entries* a_{ij} *where* $i = j$, *that mean* $(a_{11}, a_{22}, ...a_{nn})$, *see [3]* ♣

Trace of a square matrix

Definition 1.15

Let $A \in \mathcal{M}_n(\mathbb{R})$, *a square matrix of order* n, *the trace of* A *denoted* $tr(A)$, *is defined to be the sum of elements on the main diagonal; see [3].*

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{nn}$$

♣

Example 1.6

$$\text{Let } A = \begin{pmatrix} 1 & -8 & 5 \\ 0 & -3 & 2 \\ 9 & 5 & 4 \end{pmatrix} \Longrightarrow tr\,(A) = 1 + (-3) + 4 = 2$$

1.3.7 Properties of trace of matrix

A and B two square matrices of the same order, then [10]:

- $tr(A + B) = tr(A) + tr(B)$.
- $tr\,(\lambda A) = \lambda tr\,(A)$.
- $tr(A) = tr(A^t)$.
- $(AB)^t = B^t A^t$

✍ **Exercise 1.3**

Classify the following matrices (and, where possible, find the trace):

$$A = \begin{pmatrix} 9 & 2 \\ 9 & 1 \\ 11 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 4 & 7 & 0 \\ 3 & 1 & 6 & 9 \\ 1 & 0 & -3 & 4 \end{pmatrix}, C = \begin{pmatrix} 7 & 8 & 0 & 1 \\ 2 & 10 & 0 & 8 \\ -8 & 12 & -3 & 1 \\ 1 & 4 & 7 & 5 \end{pmatrix}.$$

Solution

$A \in \mathcal{M}_{3\times2}(\mathbb{R})$, $B \in \mathcal{M}_{3\times4}(\mathbb{R})$, $C \in \mathcal{M}_4(\mathbb{R})$

The trace is not defined for A or B. However, $tr(C) = 7 + 10 + (-3) + 5 = 19$

1.4 The inverse of a matrix

Definition 1.16

Let $A \in \mathcal{M}_n(\mathbb{R})$ a square matrix, if there exists a square matrix B of order n; $(B \in \mathcal{M}_n(\mathbb{R}))$ such that

$$AB = BA = I_n$$

we say that A is invertible. We call B the inverse of A and we denote it A^{-1} [6, 10] ♣

Example 1.7

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, To study if A is invertible is to study the existence of a matrix

$B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ such that $AB = BA = I_2$

AB = I is equivalent to:

$$(AB = I_2) \iff \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\iff \begin{pmatrix} x+2z & y+2t \\ z & 3t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} x + 2z = 1 \\ y + 2t = 0 \\ z = 0 \\ 3t = 1 \end{cases}$$

$$x = 1,\ y = -\frac{2}{3},\ z = 0,\ t = \frac{1}{3}$$

There is therefore only one possible matrix, namely $B = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$;

To prove that it is suitable, we must also show the equality $BA = I_2$. The matrix A is therefore invertible and $A^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$.

Remark

Not all square matrices have an inverse matrix.

Example 1.8

The matrix $A = \begin{pmatrix} 0 & 2 \\ 0 & 7 \end{pmatrix}$, is not invertible. Indeed, let $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ any matrix. So the product $BA = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 2x+7y \\ 0 & 2z+7t \end{pmatrix}$, can never be equal to the identity matrix.

Remark

- The zero matrix O_n of order n is not invertible.
- **Inverse of the inverse** If A be an invertible matrix, then A^{-1} is also invertible and we have:

$$\left(A^{-1}\right)^{-1} = A$$

- **Inverse of product** If A and B be two invertible matrices of the same order, then AB is invertible and [6]

$$(AB)^{-1} = B^{-1}A^{-1}$$

✍ **Exercise 1.4**

$$\text{Let } C = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

1. Compute $C^2 - 2C - 8I_3$
2. From the previous relation, prove that C is revertible and find its inverse.

Solution

$$C^2 = C \times C = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}$$

- *Calculate :* $C^2 - 2C - 8I_3$

$$C^2 - 2C - 8I_3 = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix} - 2\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- *Proof that* C *is invertible and find its inverse*

$$C^2 - 2C - 8I_3 = 0 \Longleftrightarrow C^2 - 2C = 8I_3$$
$$\Longleftrightarrow \begin{cases} C \cdot \frac{1}{8}(C - 2I_3) = I_3 \\ \frac{1}{8}(C - 2I_3) \cdot C = I_3 \end{cases}$$

According to the definition of the inverse of a matrix, we can conclude that C *is revertible and its inverse is given by*

$$C^{-1} = \frac{1}{8}(C - 2I_3)$$
$$= \frac{1}{8}\left[\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$C^{-1} = \frac{1}{8}\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}$$

Thus

$$C^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

1.4.1 Computation of the Matrix Inverse

1.4.1.1 Inverse of a Matrix using Elementary Row Operations (Gauss-Jordan method)

Steps to find the inverse of a matrix using Gauss-Jordan method

[19] Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix.

In order to find the inverse of the matrix following steps need to be followed [1, 2]:

Form the augmented matrix by the identity matrix $(A\,|I_n\,)$.

On the rows of this augmented matrix, we carry out elementary operations until we obtain the matrix $(I_n\,|B\,)$

The following row operations are performed on augmented matrix when required:

- $\alpha r_i \longrightarrow r_i$, with $\alpha \neq 0$: multiply each element in a row by a non-zero constant α
- $r_i + \alpha r_j \longrightarrow r_i$, with $\alpha \in \mathbb{R}$ and $i \neq j$: replace a row by the sum of itself and a constant multiple of another row of the matrix.
- $r_i \longleftrightarrow r_j$ interchange any two row (swap rows).

Example 1.9

Using elementary row operations, find A^{-1} for the matrix

$$A = \begin{pmatrix} -5 & 0 & -12 \\ 3 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

Solution

The identity matrix is given by

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Form the augmented matrix by the identity matrix $(A\,|I_n\,)$ *as follows:*

$$(A\,|I_n) = \left(\begin{array}{ccc|ccc} -5 & 0 & -12 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

We use the row operation $r_1 \leftrightarrow r_3$ to give

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ -5 & 0 & -12 & 1 & 0 & 0 \end{array}\right).$$

The pivot in the second row can be turned into a zero entry by use of the row operation $r_2 - 3r_1 \longrightarrow r_2$, giving

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -10 & 0 & 1 & -3 \\ -5 & 0 & -12 & 1 & 0 & 0 \end{array}\right).$$

A similar row operation can be applied to the third row. The row operation $r_3 + 5r_1 \longrightarrow r_3$ thus obtains:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -10 & 0 & 1 & -3 \\ 0 & 0 & 3 & 1 & 0 & 5 \end{array}\right).$$

At this point, we choose to change the new pivot in the third row, so that it is equal to 1. We use the row operation $\frac{1}{3}r_3 \longrightarrow r_3$ to find:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & -10 & 0 & 1 & -3 \\ 0 & 0 & 1 & \frac{1}{3} & 0 & \frac{5}{3} \end{array}\right).$$

To obtain the identity matrix on the left side, we need to remove the two nonzero entries which are above the pivot in the third row. The row operations $r_2 + 10r_3 \longrightarrow r_2$ and $r_1 - 3r_3 \longrightarrow r_1$ give

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -4 \\ 0 & 1 & 0 & \frac{10}{3} & 1 & \frac{41}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 0 & \frac{5}{3} \end{array}\right).$$

We have obtained precisely the form that we were looking for, which means that the right side of the augmented matrix is the inverse

$$A^{-1} = \begin{pmatrix} -1 & 0 & -4 \\ \frac{10}{3} & 1 & \frac{41}{3} \\ \frac{1}{3} & 0 & \frac{5}{3} \end{pmatrix}.$$

1.4.2 Rank of a matrix using elementary transformations

Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be a matrix having columns $C_1, C_2, ..., C_m$ and rows $R_1, R_2, ..., R_n$.

We can use elementary row/column transformations and convert the matrix into upper triangular form or in lower triangular form.

A row (or column) transformation can be one of the following: [26]

- Interchanging two rows $R_i \longleftrightarrow R_j$.
- Multiplying a row by a non-zero scalar $\alpha R_i \longrightarrow R_i$, with $\alpha \neq 0$.
- Multiplying a row by a scalar and then adding it to the other row $R_i + \alpha R_j \longrightarrow R_i$, with $\alpha \in \mathbb{R}$ and $i \neq j$.

Equivalent Matrix

[18] A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by forming finitely many successive elementary transformations on a matrix A. Denoted by $A \sim B$.

Here are the steps to find the rank of a matrix [21].

- Convert the matrix into upper triangular form or in lower triangular form using row/column transformations.
- Then the rank of the matrix is equal to the number of non-zero rows in the resultant matrix.

Example 1.10 Find the rank of each of the folowing matrices:

$$A = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ -3 & -3 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{pmatrix}, \ D = \begin{pmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{pmatrix}$$

Solution

Performing elementary row operations, we get

1. $A = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{pmatrix} \xrightarrow[R_3 - 2R_1 \longrightarrow R_3]{R_2 + R_1 \longrightarrow R_2} \begin{pmatrix} -1 & 2 & 5 \\ 0 & 4 & 8 \\ 0 & 4 & -9 \end{pmatrix} \xrightarrow{R_3 - R_2 \longrightarrow R_3} \begin{pmatrix} -1 & 2 & 5 \\ 0 & 4 & 8 \\ 0 & 0 & -17 \end{pmatrix}$

   The last equivalent matrix is in row-echelon form. It has three non-zero rows. So, $RK(A) = 3$.

2. $B = \begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 4 & 4 & -8 \end{pmatrix} \xrightarrow[R_3 - \frac{4}{3}R_1 \longrightarrow R_3]{R_2 - \frac{1}{3}R_1 \longrightarrow R_2} \begin{pmatrix} 3 & 2 & -6 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{4}{3} & 0 \end{pmatrix} \xrightarrow{R_3 - 4R_2 \longrightarrow R_3} \begin{pmatrix} 3 & 2 & -6 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

   Now it is in Echelon form and so now we have to count the number of non-zero rows.

   The number of non-zero rows is 2. *Therefore,* $RK(B) = 2$.

3. $C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{pmatrix} \xrightarrow[R_3 + 3R_1 \longrightarrow R_3]{R_2 - 2R_1 \longrightarrow R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

   The number of non-zero rows is 1. *Therefore,* $RK(C) = 1$.

4. $D = \begin{pmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{pmatrix} \xrightarrow[R_3 - \frac{3}{2}R_1 \longrightarrow R_3]{R_2 + \frac{3}{4}R_1 \longrightarrow R_2} \begin{pmatrix} 4 & 3 & 1 & -2 \\ 0 & \frac{5}{4} & \frac{-5}{4} & \frac{5}{2} \\ 0 & \frac{5}{2} & \frac{-5}{2} & 5 \end{pmatrix}$

$$\xrightarrow{R_3 - 2R_2 \longrightarrow R_3} \begin{pmatrix} 4 & 3 & 1 & -2 \\ 0 & \frac{5}{4} & \frac{-5}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The number of non-zero rows is 2. *Therefore,* $RK(D) = 2$.

1.5 Matrix associated with a linear application

[30] A linear application (or linear transformation, linear map) between two finite-dimensional vector spaces can always be represented by a matrix, called the matrix of the linear map.

Let U and V be two vector spaces over a field k such: $\dim U = n$, $\dim V = m$

Let $B = \{u_1, u_2, ..., u_n\}$ is a basis of vector space U and $B' = \{v_1, v_2, ..., v_m\}$ is a basis of vector space V

Let f be a linear transformation (map) from U to V

$$\begin{cases} f(u_1) = a_{11}v_1 + a_{21}v_2 + ... + a_{m1}v_m \\ f(u_2) = a_{12}v_1 + a_{22}v_2 + ... + a_{m2}v_m \\ \vdots \\ f(u_n) = a_{1n}v_1 + a_{2n}v_2 + ... + a_{mn}v_m \end{cases}$$

Then, the matrix defined by $A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \in \mathcal{M}_{m,n}(k)$ is the matrix associated of the linear application f and is denoted by $\mathcal{M}_f(B, B')$.

Remark

- If $\dim U = \dim V = n$, then the associated matrix of linear map will be a square matrix of order n.
- The matrix associated with a null map is the null matrix.
- The matrix associated with an identity application is the identity Matrix matrix.

Example 1.11

Let f be a linear map defined by

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x, y) \mapsto (5x + y, -x + 3y, x - y)$$

Find the associate matrix of linear map f with canonical basis in $\mathbb{R}^2$ and $\mathbb{R}^3$

Solution

The canonical basis in $\mathbb{R}^2$ *is* $\{(1,0), \ (0,1)\}$,

and the canonical basis in $\mathbb{R}^3$ *is* $\{(1,0,0), \ (0,1,0), \ (0,0,1)\}$

We have

$$\begin{aligned} f(1,0) &= (5,-1,1) = 5(1,0,0) - 1(0,1,0) + 1(0,0,1) \\ f(0,1) &= (1,3,-1) = 1(1,0,0) + 3(0,1,0) - 1(0,0,1) \end{aligned}$$

Then the associated matrix of f *is defined by*

$$A = \begin{pmatrix} 5 & 1 \\ -1 & 3 \\ 1 & -1 \end{pmatrix} \in \mathcal{M}_{3\times 2}(\mathbb{R})$$

Remark

- The associated matrix A of a linear map f is invertible if and only if the transformation f is bijective
- Rank$\left(\mathcal{M}_f\left(B, B'\right)\right) = \dim(Imf)$

1.6 Linear application (map) associated with a matrix

Definition 1.17

[27] Let $M = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$ *be a matrix of order* $m \times n$ *and let* U *and* V *be two vector spaces such that* $\dim U = n$, $\dim V = m$.

Let $B = \{u_1, u_2, ..., u_n\}$ *is a basis of* U *and* $B' = B = \{v_1, v_2, ..., v_m\}$ *is a basis of* V *,*

We call the application linear $f : U \longrightarrow V$ *an associated linear application of the matrix* M *if*

$$f(u_j) = a_{1j}v_1 + a_{2j}v_2 + ... + a_{mj}v_m \ \textit{for all } 1 \le j \le n$$

It is denoted by f_M. ♣

Example 1.12

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, Let $M = \begin{pmatrix} 4 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix}$ be a matrix of order 3×2

Let $B = \{e_1,\ e_2\}$ be the canonical basis in $\mathbb{R}^2$, and $B^{'} = \{e_1, e_2,\ e_3\}$ is the canonical basis in $\mathbb{R}^3$

There existe an unique linear application $f_M : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such

$$\begin{cases} f_M(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 = (4, -1, 2) \\ f_M(e_2) = a_{21}e_1 + a_{22}e_2 + a_{23}e_3 = (0, 1, 3) \end{cases}$$

Let $(x, y) \in \mathbb{R}^2$

$$\begin{aligned} f_M(x, y) &= f_M(xe_1 + ye_2) \\ &= xf_M(e_1) + yf_M(e_2) \\ &= x(4, -1, 2) + y(0, 1, 3) \\ &= (4x, -x + y, 2x + 3y) \end{aligned}$$

Remark

- If f is bijective, and $\mathcal{M}_f\left(B, B^{'}\right)$ is the associate matrix of f , then $\mathcal{M}_{f^{-1}}\left(B^{'}, B\right)$ is the associate matrix of f^{-1} in the basis $\left(B^{'}, B\right)$
- Let $\mathcal{M}_f$ is the associate matrix of f and $\mathcal{M}_g$ is the associate matrix of g, then $\mathcal{M}_{f\circ g} = \mathcal{M}_g \cdot \mathcal{M}_f$

1.7 Change of basis (Transit matrix)

The change of basis is a technique that allows us to express vector coordinates with respect to a ”new basis” that is different from the ”old basis” originally employed to compute coordinates.

Property

If V is a vector space with basis $\{u_1, u_2, ..., u_n\}$, then every vector $v \in V$ can be written uniquely as a linear combination of $u_1, u_2, ..., u_n$.

Definition 1.18

[4] Let V be a vector space. Let $B = \{u_1, u_2, ..., u_n\}$ and $B' = \{v_1, v_2, ..., v_n\}$ be two basis for U. Then, there exists a matrix P, denoted by $P_{B\to B'}$ and called change-of-basis matrix from B to B', such that, $P = (a_{ij}) \in M_n(k)$ where $(a_{ij}) \in k$ defined by:

$$\begin{cases} u_1 = a_{11}v_1 + a_{12}v_2 + ... + a_{1n}v_n \\ u_2 = a_{21}v_1 + a_{22}v_2 + ... + a_{2n}v_n \\ \vdots \\ u_n = a_{n1}v_1 + a_{n2}v_2 + ... + a_{nn}v_n \end{cases}$$

♣

Example 1.13

Consider the vector space $\mathbb{R}^2$ of two basis: $B = \{u_1 = (1, 0), u_2 = (1, -1)\}$ and $B' = \{v_1 = (0, 1), v_2 = (1, 1)\}$

We have

$$\begin{cases} u_1 = -v_1 + v_2 \\ u_1 = -2v_2 + v_2 \end{cases}$$

the change-of-basis matrix is

$$P_{B\to B'} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

1.8 Determinants

1.9 Determinant of a Square Matrix

To define the determinant of a matrix, we have to know about sub-matrices of a matrix.

Definition 1.19 (Sub-matrix)

Let $A = (a_{ij}) \in \mathcal{M}_{m\times n}(\mathbb{R})$ be matrix of order $m \times n$. The matrix A_{ij} obtained by deleting the i^{th} row and j^{th} column of A is called a ***sub-matrix*** *of A.*

♣

Definition 1.20 (Determinant)

[4] Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a square matrix of order n.

The determinant of the matrix A can be defined as a linear application defined from the set of square matrices $\mathcal{M}_n(\mathbb{R})$ to $\mathbb{R}$, written $det(A)$ or $|A|$ given by:

$$\begin{array}{rccl} \det: & \mathcal{M}_n(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & A & \mapsto & \det(A) = \begin{cases} a, \text{ if } A = (a), \ (n=1) \\ \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}), \text{ if } n \geq 2 \end{cases} \end{array}$$

where A_{1j} is the sub-matrix of A, which is obtained by deleting the first row and j^{th} column. ♣

Remark

- The determinant of the matrix A can be calculated according to any row i and according to any column j, it is given by the following relations:
  - according to the row $i \Longrightarrow \det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$
  - according to the column $j \Longrightarrow \det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$

where A_{ij} is a sub-matrix of A.

Definition 1.21 (Minor and Cofactor)

[4] Let A_{ij} a square sub-matrix of $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$.

1. *The determinant $M_{ij} = \det(A_{ij})$ is called the* ***minor*** *of the element a_{ij} of A.*
2. *$C_{ij} = (-1)^{i+j} M_{ij}$ is called the* ***cofactor*** *of a_{ij}.* ♣

So, we can defined the determinant of a matrix A using minors and cofactors as:

$$\det(A) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Example 1.14

- Let $A = \begin{pmatrix} 3 & 5 \\ 4 & -2 \end{pmatrix}$, So $\det A = 3 \times (-2) - 4 \times 5 = -26$

- Let $A = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{pmatrix}$, we calculate the determinant of the matrix A according to the first row

$$\begin{aligned} \begin{vmatrix} -1^{+} & 2^{-} & 5^{+} \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{vmatrix} &= +(-1)\begin{vmatrix} 2 & 3 \\ 8 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ -2 & 8 \end{vmatrix} \\ &= (-1)(2 \times 1 - 3 \times 8) - 2(1 \times 1 - (-2) \times 3) + 5(1 \times 8 - (-2)\,2) \\ &= 68 \end{aligned}$$

We calculate the determinant of the matrix A according to the second column

$$\begin{aligned} \begin{vmatrix} -1 & -2 & 5 \\ 1 & +2 & 3 \\ -2 & -8 & 1 \end{vmatrix} &= -2\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} -1 & 5 \\ -2 & 1 \end{vmatrix} - 8\begin{vmatrix} -1 & 5 \\ 1 & 3 \end{vmatrix} \\ &= -2(1 \times 1 - (-2) \times 3) + 2((-1) \times 1 - (-2) \times 5) - 8((-1) \times 3 - 1 \times 5) \\ &= 68 \end{aligned}$$

Remark

To facilitate the calculations, we have to choose the row or column that contains the largest number of zeros.

Example 1.15

Calculate the determinant of the following matrix

$$B = \begin{pmatrix} 3 & -7 & 9 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 10 & 3 \\ 0 & 4 & -8 & 5 \end{pmatrix}$$

We calculate the determinant of the matrix A according to the first column (it contains the largest number of zeros.)

$$\begin{aligned}\det B &= \begin{vmatrix} 3 & -7 & 9 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 10 & 3 \\ 0 & 4 & -8 & 5 \end{vmatrix} = 3 \begin{vmatrix} 0 & 2 & 0 \\ 1 & 10 & 3 \\ 4 & -8 & 5 \end{vmatrix} \\ &= 3 \times (-2) \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = (-6)\,(1 \times 5 - 3 \times 4) \\ &= 42\end{aligned}$$

1.9.1 Important Properties of Determinants

[4, 29] Let $A \in \mathcal{M}_n(\mathbb{k})$ be a matrix having columns $C_1, C_2, ..., C_n$ and rows $R_1, R_2, ..., R_n$.

1. If all the elements of a row (or column) are zero, then $\det A = 0$.
2. The interchange of any two rows (or columns) of the determinant of A changes its sign.

   Example 1.16

   $$|A| = \begin{vmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{vmatrix}$$

   $$|B| = \begin{vmatrix} 2 & -1 & 5 \\ 2 & 1 & 3 \\ 8 & -2 & 1 \end{vmatrix} \text{(interchange of columns 1 and 2)}$$

   $$|C| = \begin{vmatrix} -2 & 8 & 1 \\ 1 & 2 & 3 \\ -1 & 2 & 5 \end{vmatrix} \text{(interchange of rows 1 and 3)}$$

   $|A| = 68$ (previous example)

   $|B| = 2\,(1+6) + (20-24) + 5\,(-4-8) = -68$

   So $|A| = -\,|B|$.

$|C| = (-2)(1-6) - 8(5+3) + 10(2+2) = -68$

So $|A| = -|C|$.

3. A determinant remains unaltered under an operation of the form

$$C_i + \sum_{j=1, j\neq i}^{n} \alpha_j C_j \longrightarrow C_i$$

Or

$$R_i + \sum_{j=1, j\neq i}^{n} \alpha_j R_j \longrightarrow R_i$$

(This property is used to make zeros appear on a row (or column))

Example 1.17

$$\text{Let: } detA = \begin{array}{c} \quad C_1 \;\; C_2 \;\; C_3 \\ \quad \downarrow \;\;\; \downarrow \;\;\; \downarrow \\ \begin{vmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{vmatrix} \end{array} \begin{array}{l} \\ \\ \longleftarrow R_1 \\ \longleftarrow R_2 \\ \longleftarrow R_3 \end{array}$$

Replacing the first column by: $C_1 + \frac{1}{2}C_2 \longrightarrow C_1$

$$\begin{vmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{vmatrix} = \begin{vmatrix} -1+1 & 2 & 5 \\ 1+1 & 2 & 3 \\ -2+4 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 5 \\ 2 & 2 & 3 \\ 2 & 8 & 1 \end{vmatrix} \begin{array}{l} \longleftarrow R_1 \\ \longleftarrow R_2 \\ \longleftarrow R_3 \end{array}$$

Replacing the second row by: $R_2 - R_3 \longrightarrow R_2$

$$\begin{vmatrix} 0 & 2 & 5 \\ 2 & 2 & 3 \\ 2 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 5 \\ 2-2 & 2-8 & 3-1 \\ 2 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 5 \\ 0 & -6 & 2 \\ 2 & 8 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 5 \\ -6 & 2 \end{vmatrix} = 2(4+30) = 68$$

4. If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.
   - Special case: $\det(\alpha A) = \alpha^n \cdot \det A$, where $A \in \mathcal{M}_n(\mathbb{k})$

Example 1.18

$$|A| = \begin{vmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{vmatrix}, |B| = \begin{vmatrix} -1 & 2 & -15 \\ 1 & 2 & -9 \\ -2 & 8 & -3 \end{vmatrix}, |C| = \begin{vmatrix} -1 & 2 & 5 \\ 2 & 4 & 6 \\ -2 & 8 & 1 \end{vmatrix}$$

$|A| = 68$ (previous example)

In $|B|$, the third column is multiplied by (-3)

$$|B| = \begin{vmatrix} -1 & 2 & -15 \\ 1 & 2 & -9 \\ -2 & 8 & -3 \end{vmatrix} = (-1)(-60+72) - 2(-3-18) - 15(8+4)$$

So, $|B| = -204$, that means $|B| = -3|A|$.

In $|C|$ the second row is multiplied by 2

$$|C| = \begin{vmatrix} -1 & 2 & 5 \\ 2 & 4 & 6 \\ -2 & 8 & 1 \end{vmatrix} = (-1)(40-48) - 2(20+12) + 5(16+8)$$

So, $|C| = 136$, that means $|C| = 2|A|$.

5. If all elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then $\det A = 0$.

Example 1.19

$$|A| = \begin{vmatrix} -1 & 2 & 4 \\ 1 & 2 & 4 \\ -2 & 8 & 16 \end{vmatrix}$$

The third column is proportional to first column

$$|A| = \begin{vmatrix} -1 & 2 & 4 \\ 1 & 2 & 4 \\ -2 & 8 & 16 \end{vmatrix} = (-1)(32-32) - 2(16+8) + 4(8+4) = 0.$$

1.9.2 Determinants of particular matrices

Let $A, B \in \mathcal{M}_n(\mathbb{k})$ two square matrices of order n, then

1. If A is a Zero-matrix, then $\det A = 0$.

2. If $A = (a_{ij})$ is an Upper triangular matrix (or Lower triangular matrix), then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot .... \cdot a_{nn}.$$

   Special cases:

   - If A is Diagonal matrix, then $\det A = a_{11} \cdot a_{22} \cdot .... \cdot a_{nn}$.
   - $\det I_n = 1$.

3. $\det A = \det A^t$

   Example 1.20

   Let

$$A = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{pmatrix}$$

   $|A| = 68$.

$$A^t = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & 8 \\ 5 & 3 & 1 \end{pmatrix}$$

$$|A^t| = (-1)\,(20-24) - (20-40) + (-2)\,(6-10) = 68$$

$$\text{So, } |A| = |A^t|$$

4. $\det AB = \det A \cdot \det B$

   - Special case: $\det AA^{-1} = \det A \cdot \det A^{-1} = 1 \Longrightarrow \det A^{-1} = \frac{1}{\det A}$

Example 1.21

$$\text{Let } A = \begin{pmatrix} -1 & 2 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

A is upper triangular matrix, then $\det A = |A| = (-1) \times 2 \times 1 = -2$.

B is diagonal matrix, then $\det B = |B| = (-1) \times 3 \times (-2) = 6$.

$\det A \times \det B = (-2) \times 6 = -12$

$$AB = \begin{pmatrix} -1 & 2 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 6 & -10 \\ 0 & 6 & -6 \\ 0 & 0 & -2 \end{pmatrix}$$

AB is upper triangular matrix, then $\det AB = |AB| = 1 \times 6 \times (-2) = -12$

$\det AB = -12$, so $\det AB = \det A \cdot \det B$.

Remark

- $\det(A + B) \neq \det(A) + \det(B)$.
- $\det(\alpha \cdot A) \neq \alpha \cdot \det(A)$.

1.9.3 Inverse of a square Matrix using Cofactor Matrix

Definition 1.22

[15] Let $A = (a_{ij})_{1 \leq i,j \leq n}$ *be a square matrix of order* n*, where* $\det A \neq 0$

We can calculate the inverse of matrix A *as follows:*

$$A^{-1} = \frac{1}{\det A} (Cof A)^t \tag{1.2}$$

$Cof A = (C_{ij})_{1 \leq i,j \leq n}$, *and* $C_{ij} = (-1)^{i+j} M_{ij}$,

Where M_{ij}*, denote the minor of the element* a_{ij}*,* C_{ij} *is the* ***cofactor*** *of* a_{ij}. ♣

Example 1.22

$$\text{Let } A = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$$

Calculate the inverse of the matrix A

According to equation (1.2) we have: $A^{-1} = \frac{1}{\det A}(CofA)^t$

$\det A = 14 \neq 0$, then A is invertible

$$CofA = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \text{ and } C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = (-1)^2 \times 4 = 4,$$

$$C_{12} = (-1)^3 \times 2 = -2,$$

$$C_{21} = (-1)^3 \times 3 = -3,$$

$$C_{22} = (-1)^4 \times 5 = 5.$$

$$CofA = \begin{pmatrix} 4 & -2 \\ -3 & 5 \end{pmatrix} \Longrightarrow (CofA)^t = \begin{pmatrix} 4 & -3 \\ -2 & 5 \end{pmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{14}\begin{pmatrix} 4 & -3 \\ -2 & 5 \end{pmatrix} \Longrightarrow A^{-1} = \begin{pmatrix} \frac{2}{7} & \frac{-3}{14} \\ \frac{-1}{7} & \frac{5}{14} \end{pmatrix}$$

Example 1.23

Calculate the inverse of the matrix

$$A = \begin{pmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{pmatrix}$$

$\det A = 54 \neq 0$, then A is invertible.

$CofA = (C_{ij})_{1 \leq i,j \leq n}$, and $C_{ij} = (-1)^{i+j} M_{ij}$

$$C_{11} = (-1)^2 \begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 22, \quad C_{12} = (-1)^3 \begin{vmatrix} 7 & 5 \\ 1 & 4 \end{vmatrix} = -23, \quad C_{13} = (-1)^4 \begin{vmatrix} 7 & 3 \\ 1 & -2 \end{vmatrix} = -17,$$

$$C_{21} = (-1)^3 \begin{vmatrix} 4 & -6 \\ -2 & 4 \end{vmatrix} = -4, \quad C_{22} = (-1)^4 \begin{vmatrix} 2 & -6 \\ 1 & 4 \end{vmatrix} = 14, \quad C_{23} = (-1)^5 \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = 8,$$

$$C_{31} = (-1)^4 \begin{vmatrix} 4 & -6 \\ 3 & 5 \end{vmatrix} = 38, \quad C_{32} = (-1)^5 \begin{vmatrix} 2 & -6 \\ 7 & 5 \end{vmatrix} = -52, \quad C_{33} = (-1)^6 \begin{vmatrix} 2 & 4 \\ 7 & 3 \end{vmatrix} = -22.$$

$$CofA = \begin{pmatrix} 22 & -23 & -17 \\ -4 & 14 & 8 \\ 38 & -52 & -22 \end{pmatrix} \Longrightarrow (CofA)^t = \begin{pmatrix} 22 & -4 & 38 \\ -23 & 14 & -52 \\ -17 & 8 & -22 \end{pmatrix}$$

$$A^{-1} = \tfrac{1}{54} \begin{pmatrix} 22 & -4 & 38 \\ -23 & 14 & -52 \\ -17 & 8 & -22 \end{pmatrix}$$

1.10 Rank of a matrix using determinant

Definition 1.23

Let A be any matrix of order $m \times n$.

A matrix A is said to be of rank r if [13]

1. *It has at-least one non-zero minor of order r*
2. *Every minor of order greater than r of A is zero.*

The rank of a matrix A is denoted by $RK(A)$. ♣

Remark

- If A is a zero matrix , then $RK(A) = 0$.
- If A is not a zero matrix , then $RK(A) \geq 1$.
- If A is a matrix of order $m \times n$, then $RK(A) \leq \min(m, n)$.
- If A is a square matrix of order n, then (A invertible)$\Longleftrightarrow (RK(A) = n)$.

1.10.1 Finding Rank of a Matrix by Minor Method

Here are the steps to find the rank of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ by the minor method.

- Find the determinant of A (if A is a square matrix $A \in \mathcal{M}_n(\mathbb{R})$). If $\det(A) \neq 0$, then the rank of $A = n$.

- If either $\det A = 0$ (in case of a square matrix) or A is a rectangular matrix, then see whether there exists any minor of maximum possible order is non-zero. If there exists such non-zero minor, then rank of A is the order of that particular minor.
- Repeat the above step if all the minors of the order considered in the above step are zeros and then try to find a non-zero minor of order that is one less than the order from the above step.

Example 1.24

Find the rank of each of the following matrices:

$$A = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ -3 & -3 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{pmatrix}, \; D = \begin{pmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{pmatrix}$$

Solution

1. *Let be* $A = \begin{pmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -2 & 8 & 1 \end{pmatrix}$, *$A$ is a square matrix of order* 3, *we compute the* $\det A$, $|A| = 68$ *(previous example).* $\det(A) \neq 0$, *then the* $RK\,(A) = 3$.
2. *Let be* $B = \begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ -3 & -3 & 6 \end{pmatrix}$, *$B$ is a square matrix of order* 3, *we compute the* $\det B$.

   $\det(B) = 0$, *then the* $RK\,(B) < 3$.

   Next consider the second-order minors of B.

   we find that the second-order minor $\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$, *then the* $RK\,(B) = 2$.

3. *Let* $C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{pmatrix}$, *$C$ is a square matrix of order* 3*, we compute the* $\det C$.

$\det(C) = 0$*, then the* $RK\ (C) < 3$.

Next consider the second-order minors of C.

$M_{11} = \begin{vmatrix} 2 & 4 \\ -3 & -6 \end{vmatrix} = 0$, $M_{12} = \begin{vmatrix} 2 & 6 \\ -3 & -9 \end{vmatrix} = 0$, $M_{13} = \begin{vmatrix} 2 & 4 \\ -3 & -6 \end{vmatrix} = 0$,

$M_{21} = \begin{vmatrix} 2 & 3 \\ -6 & -9 \end{vmatrix} = 0$, $M_{22} = \begin{vmatrix} 1 & 3 \\ -3 & -9 \end{vmatrix} = 0$, $M_{23} = \begin{vmatrix} 1 & 2 \\ -3 & -6 \end{vmatrix} = 0$,

$M_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$, $M_{32} = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$, $M_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$,

All the second-order minors of C *are zero, and* C *is not a zero matrix , then* $RK\ (C) = 1$.

4. *Let* $D = \begin{pmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{pmatrix}$, *$D$ is a matrix order* 3×4*. So* $RK\ (D) \le \min(3, 4) = 3$.

We search for non-zero third-order minor of D

We have

$\begin{vmatrix} 4 & 3 & 1 \\ -3 & -1 & -2 \\ 6 & 7 & -1 \end{vmatrix} = 0$, $\begin{vmatrix} 4 & 3 & -2 \\ -3 & -1 & 4 \\ 6 & 7 & 2 \end{vmatrix} = 0$, $\begin{vmatrix} 4 & 1 & -2 \\ -3 & -2 & 4 \\ 6 & -1 & 2 \end{vmatrix} = 0$, $\begin{vmatrix} 3 & 1 & -2 \\ -1 & -2 & 4 \\ 7 & -1 & 2 \end{vmatrix} = 0$.

So, $RK\ (D) < 3$*. Next, we search for a non-zero second-order minor of* D.

We find that $\begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} = 10 \neq 0$*. So* $RK\ (D) = 2$.

Chapter 1 Exercise

1. **Exercise 1**

We consider the following matrices:

$$A = \begin{pmatrix} -1 & 4 & 7 \end{pmatrix},\ B = \begin{pmatrix} 3 \\ -1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \\ -4 & 1 \\ -1 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix},\ H = \begin{pmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

(a). What are the possible matrix products? What are square matrices and symmetric matrices?

(b). Calculate $\frac{1}{3}C, C + 2C, C.D, D^2$.

2. **Exercise 2**

Let A be a matrix defined by:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ -1 & 0 & -2 \end{pmatrix}$$

(a). Calculate $A^3 + A^2 + A$.

(b). Express A^{-1} in terms of A^2, A and I_3. Detremine A^{-1}.

3. **Exercice 3**

Let A be a matrix defined by:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a). Find $a, b \in \mathbb{R}$ such that $A^2 = a.I_3 + b.A$.

(b). Deduce that A is invertible and give its inverse..

4. **Exercise 4**

Calculate the determinants

$$\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}, \begin{vmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 1 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 8 & -1 & -2 & 0 \\ 6 & 0 & 1 & 4 \end{vmatrix}$$

5. **Exercise 5**

   Prove using the properties of determinants that:

$$\begin{vmatrix} 2 & 3 & -5 \\ 2\alpha & 3\alpha & -5\alpha \\ 4 & -1 & 8 \end{vmatrix} = 0, \begin{vmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \lambda \\ -\beta & -\lambda & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \delta & \delta^2 \end{vmatrix} = (\beta - \alpha)(\delta - \alpha)(\delta - \beta)$$

6. **Exercise 6**

   Calculate the determinants of the following matrices:

$$A = \begin{pmatrix} 5 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 \\ -6 & 4 & 2 & -8 \\ 1 & -2 & -3 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 8 & -1 & -2 & 0 \\ 6 & 0 & 1 & 4 \end{pmatrix} \times \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

Chapter 2 System of Linear Equations

Introduction

- ❏ *Introduction*
- ❏ *Study of the solution set*
- ❏ *Matrix form of linear equations*
- ❏ *Methods to solve systems*
  - *Cramer‘s rule*
  - *Matrix Inversion method*
- ❏ *Gauss method*
  - *Equivalent Systems*
  - *Elementary Operations*
  - *Solving systems using Gauss elimination*

2.1 Introduction

Systems of linear equations and their solutions constitute one of the major topics that we will study in this chapter. In the first section we will introduce some basic terminology and discuss a methods for solving such systems as Cramer's rule, Matrix inversion method, and Gauss method.

An equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n = b \tag{2.1}$$

is called a linear equation in the n unknowns (variables) $x_1, x_2, x_3, ..., x_n$.

$a_1, a_2, a_3, ..., a_n$ denote real numbers (called the coefficients of $x_1, x_2, x_3, ..., x_n$ respectively).

b is a number (called the constant term of the equation).

Example 2.1

$3x + 4y = 2,$

$x - y - z = -6.$

are linear equations, but

$3y + yz = 3$

$sin(2x) - cos(3y) = 2$ are not.

2.2 Study of the solution set

A vector $(s_1, s_2, ..., s_n)$ is called a solution of this equation if it satisfies the equation (2.1)

That means

$$a_1s_1 + a_2s_2 + a_3s_3 + ... + a_ns_n = b$$

The set of all such solutions is called the **solution set** for the equation.

m linear equations in n unknowns $x_1, x_2, x_3, ..., x_n$ of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m \end{cases} \tag{2.2}$$

is called a **system of linear equations**

We abbreviate this system by

$$\sum_{j=1}^{n} a_{ij}x_j = b_i,\ i = \overline{1..., m}$$

where a_{ij} and b_i are all real numbers.

Such a linear system is called an **homogeneous** linear system if

$$b_1 = b_2 = b_3 = ... = b_m = 0.$$

If $(s_1, s_2, ..., s_n)$ is a solution of the above system of equations, then it is a solution of each of the m equations in the system.

The set of all solutions of the linear system is called the **solution set** of the **system**. To solve a system is to find its solution set.

Remark Any system of linear equations has one of the following states [25].

- No solution.
- Unique solution.
- Infinitely many solutions.

Example 2.2

Consider the system two equations in two Variables

$$\begin{cases} 2x - y = 1 \\ 3x + 2y = 12 \end{cases}$$

The **unique solution of the system** is given by $x = 2$ and $y = 3$. Geometrically, the two lines represented by the two linear equations that make up the system intersect at the point $(2, 3)$. See Figure 2.1

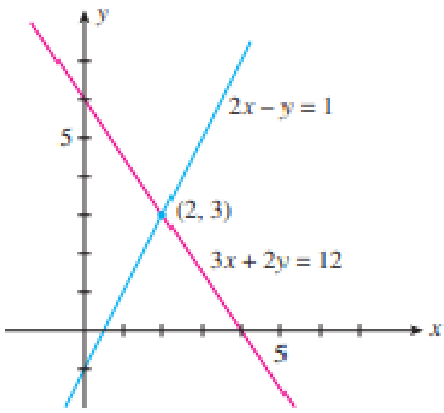


Figure 2.1: Unique solution

Example 2.3

Consider the system

$$\begin{cases} 2x - y = 1 \\ 6x - 3y = 3 \end{cases}$$

The system of two equations is equivalent to the single equation $2x - y = 1$. Thus, any ordered pair of numbers (x, y) satisfying the equation $2x - y = 1$ constitutes a solution to the system.

So, there are **infinitely solutions of the system**. Geometrically, the two equations in the system represent the same line, and all solutions of the system are points lying on the line see Figure 2.2.

Example 2.4

Consider the system

$$\begin{cases} 2x - y = 1 \\ 6x - 3y = 12 \end{cases}$$

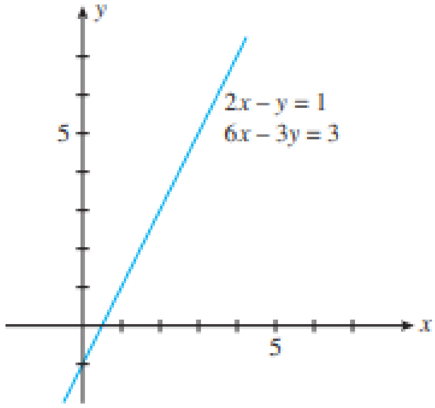


Figure 2.2: Infinitely solutions

By the first equation, we obtain the equation $y = 2x - 1$, substituting this expression into the second equation gives $0 = 9$.

which is impossible. Thus, there is **no solution to the system of equations**.

We see at once that the lines represented by these equations are parallel see Figure 2.3.

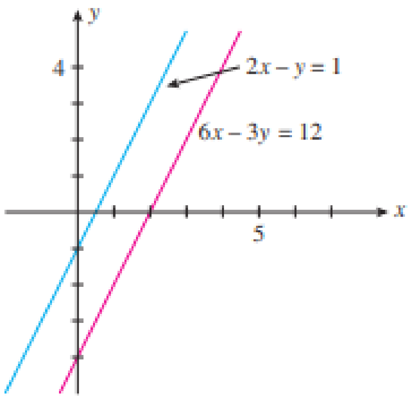


Figure 2.3: No solution

Definition 2.1

[25] We say that the system of linear equations is ***consistent*** *if it has a solution (unique solution or infinitely many solutions). Otherwise the system is called* ***inconsistent*** *(no solution).* ♣

2.3 Matrix representation of linear equations

Consider a system of m linear equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m \end{cases} \tag{2.3}$$

We can write the system of equations (2.3) in matrix form as:

$$AX = B$$

Where $A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ comprised of the coefficients of the variables,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

The n unknowns (variables), is written in a single column $X = (x_j)\,, j = \overline{1...,n}$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The constant matrix $B = (b_i)\,, i = \overline{1...,m}$ of order $m \times 1$ is written in a single column and in the same order as the rows of the coefficient matrix.

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Remark

The matrix form of homogeneous linear equations is $AX = 0$ called the associated homogeneous system.

The augmented matrix is the coefficient matrix with the constant matrix as the last column, ie. $[A\,|B]$

$$[A\,|B] = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

2.4 Methods to solve systems of linear equations

The following methods are useful to solve linear equation

1. Cramer‘s rule
2. Matrix Inversion method
3. Gauss elimination method

Remark

The first two methods (Cramer‘s rule and Matrix Inversion methods) are applicable only when $m = n$ i.e. to solve system of n equations in n unknowns.

2.4.1 Cramer‘s rule

Definition 2.2 (Cramer‘s rule)

[20, 24]Let $AX = B$ be a linear system with n equations in n unknowns, where:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

If $|A| \neq 0$*, then the unique solution to this system is*

$$X = (x_j)\,,\ j = \overline{1...,n}\,,\ \text{and } x_j \text{ is given by: } x_j = \frac{|A_j|}{|A|},\ j = \overline{1...,n}$$

where A_j *is the matrix obtained from* A *by replacing the* j^{th} *column of A by the column vector* B*, in other words*

$$A_j = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj-1} & b_m & a_{mj+1} & \cdots & a_{mn} \end{pmatrix}$$

♣

Example 2.5

Solve the following system by using Cramer's Rule

$$\begin{cases} x - y + z = -8 \\ 3x + y - 2z = -12 \\ 2x + 3y - 2z = 8 \end{cases}$$

Solution

The given system can be written in the matrix form $AX = B$

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & -2 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 \\ -12 \\ 8 \end{pmatrix}$$

We start with the coefficient determinant

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & -2 \\ 2 & 3 & -2 \end{vmatrix} = 1 \begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} - (-1) \begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \\ &= (-2+6) + (-6+4) + (9-2) \\ &= 9 \end{aligned}$$

$|A| \neq 0$*, then the unique solution to this system is given by:* $x_j = \frac{|A_j|}{|A|}$, $j = 1, 2, 3.$

Now we compute the other determinants $|A_j|$

$$\begin{aligned}|A_1| = \begin{vmatrix} -8 & -1 & 1 \\ -12 & 1 & -2 \\ 8 & 3 & -2 \end{vmatrix} &= -8\begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} - (-1)\begin{vmatrix} -12 & -2 \\ 8 & -2 \end{vmatrix} + 1\begin{vmatrix} -12 & 1 \\ 8 & 3 \end{vmatrix} \\ &= -8(-2+6) + (24+16) + (-36-8) \\ &= -32 + 40 - 44 \\ &= -36\end{aligned}$$

$$\begin{aligned}|A_2| = \begin{vmatrix} 1 & -8 & 1 \\ 3 & -12 & -2 \\ 2 & 8 & -2 \end{vmatrix} &= 1\begin{vmatrix} -12 & -2 \\ 8 & -2 \end{vmatrix} - (-8)\begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & -12 \\ 2 & 8 \end{vmatrix} \\ &= (24+16) + 8(-6+4) + (24+24) \\ &= 40 - 16 + 48 \\ &= 72\end{aligned}$$

$$\begin{aligned}|A_3| = \begin{vmatrix} 1 & -1 & -8 \\ 3 & 1 & -12 \\ 2 & 3 & 8 \end{vmatrix} &= 1\begin{vmatrix} 1 & -12 \\ 3 & 8 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -12 \\ 2 & 8 \end{vmatrix} + (-8)\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \\ &= (8+36) + (24+24) - 8(9-2) \\ &= 44 + 48 - 56 \\ &= 36\end{aligned}$$

Now, the solutions given by the formulas

$$x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad z = \frac{|A_3|}{|A|}$$

$$x = \frac{-36}{9} = -4, \quad y = \frac{72}{9} = 8, \quad z = \frac{18}{36} = \frac{1}{2}.$$

2.4.2 Matrix Inversion method

Definition 2.3

[20, 24] Let $AX = B$ be a non-homogeneous linear system with n equations in n unknowns,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

If $|A| \neq 0$, then the system has unique solution given by

$$X = A^{-1}B$$

♣

Example 2.6 Solve the system of equations using matrix inverses

$$\begin{cases} x - 2y + z = 3 \\ 2x + y + -z = 5 \\ 3x - y + 2z = 12 \end{cases}$$

The given system can be written in the matrix form $AX = B$

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 15 \end{pmatrix}$$

We start with the coefficient determinant

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ &= 10 \end{aligned}$$

$|A| = 10 \neq 0$. Hence A^{-1} exists and the system has unique solution given by

$$X = A^{-1}B$$

Find A^{-1} using cofactor matrix

$$A^{-1} = \frac{1}{|A|} (CofA)^t$$

$$CofA = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ +\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$CofA = \begin{pmatrix} 1 & -7 & -5 \\ 3 & -1 & -5 \\ 1 & 3 & 5 \end{pmatrix}$$

Now transpose this matrix and divide by $|A|$ to obtain A^{-1}.

So,

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{pmatrix}$$

Finally $X = A^{-1}B$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ \frac{-7}{10} & \frac{-1}{10} & \frac{3}{10} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{10} \times 3 + \frac{3}{10} \times 5 + \frac{1}{10} \times 12 \\ \frac{-7}{10} \times 3 - \frac{1}{10} \times 5 + \frac{3}{10} \times 12 \\ \frac{-1}{2} \times 3 - \frac{1}{2} \times 5 + \frac{1}{2} \times 12 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

Then the unique solution is given by $x = 3$, $y = 1$, $z = 2$.

Remark

Let $AX = 0$ be an homogeneous linear system with n equations in n unknowns,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

If $|A| \neq 0$, then the system has unique solution given by

$$x_1 = 0, \ x_2 = 0, x_3 = 0, \ ......, x_n = 0$$

This solution is called trivial solution.

2.5 Gauss method

The Gauss method will apply to linear systems of any size, including systems where the number of equations and the number of variables are not the same.

To solving systems of linear equations of any size, we write a series of systems, one after the other, each equivalent to the previous system.

Each of these systems has the same set of solutions as the original one; the aim is to end up with a system that is easy to solve.

2.5.1 Equivalent systems and elementary operations

Definition 2.4

[20, 24] Two systems of equations are equivalent if they have the same solution set. ♣

Remark (**Elementary operations that produce equivalent systems**])

A system of linear equations is transformed into an equivalent system if

(a) Two equations are interchanged.

(b) An equation is multiplied by a non-zero constant.

(c) A constant multiple of one equation is added to another equation.

2.5.2 Solving linear systems using augmented matrices (Gauss elimination)

[20, 24] Consider a system of m linear equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m \end{cases} \tag{2.4}$$

We use rank of matrices to determine consistency or inconsistency of a system.

2.5.3 Steps of the Gauss elimination method

Find the ranks of the coefficient matrix A and augmented matrix $(A\,|B\,)$ for which,

Reduce the augmented matrix $(A\,|B\,)$ by elementary row operations on matrices to get the upper triangular form.

This form gives the rank of the augmented matrix $(A\,|B\,)$ and also the rank of A.

1. If rank$(A) \neq$ rank$(A\,|B\,)$, then the system has no solution.
2. If rank(A) =rank$(A\,|B\,)$ = n, then the system has a unique solution.
3. If rank(A) =rank$(A\,|B\,) < n$, then the system has infinitely many solutions.

Remark

An $m \times n$ homogeneous system $AX = 0$ has

- infinitely many solutions if rank$(A) < n$,
- unique (trivial) solution if rank$(A) = n$.

Example 2.7

Solve the following system of equations.

$$\begin{cases} -4x + 8y - z = -12 \\ x - 3y + z = 10 \\ 3x - 7y + 2z = 24 \end{cases} \tag{2.5}$$

Solution

We can write the linear system (2.5) in matrix form as

$$\begin{pmatrix} -4 & 8 & -1 \\ 1 & -3 & 1 \\ 3 & -7 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 \\ 10 \\ 24 \end{pmatrix}$$

We create the augmented matrix

$$(A\,|B\,) = \left(\begin{array}{ccc|c} -4 & 8 & -1 & -12 \\ 1 & -3 & 1 & 10 \\ 3 & -7 & 2 & 24 \end{array}\right)$$

Use the elementary row operations to reduce the augmented matrix $(A\,|B\,)$

$$\left(\begin{array}{ccc|c} -4 & 8 & -1 & -12 \\ 1 & -3 & 1 & 10 \\ 3 & -7 & 2 & 24 \end{array}\right) \begin{array}{c} R_1 \longleftrightarrow R_2 \\ \sim \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 10 \\ -4 & 8 & -1 & -12 \\ 3 & -7 & 2 & 24 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 10 \\ -4 & 8 & -1 & -12 \\ 3 & -7 & 2 & 24 \end{array}\right) \begin{array}{c} R_2 + 4R_1 \longrightarrow R_2 \\ \sim \\ R_3 - 3R_1 \longrightarrow R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 10 \\ 0 & -4 & 3 & 28 \\ 0 & 2 & -1 & -6 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 10 \\ 0 & -4 & 3 & 28 \\ 0 & 2 & -1 & -6 \end{array}\right) \begin{array}{c} \\ \sim \\ R_3 + \frac{1}{2}R_2 \longrightarrow R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 10 \\ 0 & -4 & 3 & 28 \\ 0 & 0 & \frac{1}{2} & 8 \end{array}\right)$$

rank(A) =*rank*$(A\,|B\,) = 3$*, then the system has unique solution*

$$\begin{cases} x - 3y + z = 10 \\ -4y + 3z = 28 \\ \frac{1}{2}z = 8 \end{cases}$$

So, the solution is given by

$$\{z = 16,\ y = 5,\ x = 9\}$$

Example 2.8

Find all solutions (if any) to the following system of linear equations.

$$\begin{cases} 3x + 3y - 2z = 1 \\ x + 2y = 4 \\ 10y + 3z = -2 \\ 4x - 6y - 2z = 10 \end{cases}$$

This is a system of 4 linear equations in 3 unknown

The given system can be written in the matrix form $AX = B$

$$\begin{pmatrix} 3 & 3 & -2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 4 & -6 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -2 \\ 10 \end{pmatrix}$$

The augmented matrix

$$(A\,|B\,) = \left(\begin{array}{ccc|c} 3 & 3 & -2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 4 & -6 & -2 & 10 \end{array}\right)$$

We use the elementary row operations to reduce the augmented matrix $(A\,|B\,)$

$$\left(\begin{array}{ccc|c} 3 & 3 & -2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & -3 & -2 \\ 4 & -6 & -2 & 10 \end{array}\right) \underset{\sim}{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & -2 & 1 \\ 0 & 10 & -3 & -2 \\ 4 & -6 & -2 & 10 \end{array}\right) \begin{array}{c} R_2 - 3R_1 \longrightarrow R_2 \\ \sim \\ R_4 - 4R_1 \longrightarrow R_4 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & -2 & -11 \\ 0 & 10 & -3 & -2 \\ 0 & -14 & -2 & -6 \end{array}\right) \begin{array}{c} R_3 + \frac{10}{3}R_2 \longrightarrow R_3 \\ \sim \\ R_4 - \frac{14}{3}R_2 \longrightarrow R_4 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & -2 & -11 \\ 0 & 0 & -\frac{29}{3} & -\frac{116}{3} \\ 0 & 0 & -\frac{34}{3} & -\frac{136}{3} \end{array}\right) \sim$$

$$\begin{matrix} \frac{-3}{29}R_3 \longrightarrow R_3 \\ \sim \\ \frac{-3}{34}R_4 \longrightarrow R_4 \end{matrix} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & -2 & -11 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \quad \begin{matrix} \sim \\ R_4 - R_3 \longrightarrow R_4 \end{matrix} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & -2 & -11 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

rank(A) =rank$(A\,|B\,)$ = 3= number of unknowns therefore the given system is consistent and

has unique solution

from the above form, the given equivalent system reduces to

$$\begin{cases} x + 2y = 4 \\ -3y - 2z = -11 \\ z = 4 \end{cases}$$

$$\Longrightarrow z = 4, y = 1, x = 2$$

Thus the unique solution is $x = 2$, $y = 1$, $z = 4$.

Example 2.9

Find all solutions (if any) to the following system of linear equations.

$$\begin{cases} 3x + y - 4z = -1 \\ x + 10z = 5 \\ 8x + 2y + 12z = 2 \end{cases}$$

The given system can be written in the matrix form $AX = B$

$$\begin{pmatrix} 3 & 1 & -4 \\ 1 & 0 & 10 \\ 8 & 2 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$$

Solution *The corresponding augmented matrix is*

$$(A\,|B\,) = \left(\begin{array}{ccc|c} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 8 & 2 & 12 & 2 \end{array}\right)$$

Create the first leading one by interchanging rows 1 and 2

$$\left(\begin{array}{ccc|c} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 8 & 2 & 12 & 2 \end{array}\right) \underset{\sim}{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 8 & 2 & 12 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 8 & 2 & 12 & 2 \end{array}\right) \begin{array}{c} R_2 - 3R_1 \longrightarrow R_2 \\ \sim \\ R_3 - 8R_1 \longrightarrow R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 2 & -68 & -38 \end{array}\right)$$

Now subtract row 2 from row 3 to obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 2 & -68 & -38 \end{array}\right) \begin{array}{c} \\ \sim \\ R_3 - 2R_2 \longrightarrow R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

rank$(A) = 2$, *but rank*$(A\,|B\,) = 3$,

rank$(A) \neq$*rank*$(A\,|B\,)$, *then the system has no solution.*

it is clear that the following reduced system

$$\begin{cases} x + 10z = 5 \\ y - 34z = -16 \\ 0 = -6 \end{cases}$$

has no solution, which is equivalent to the original system. Hence the original system has no solution.

Example 2.10

Find all solutions (if any) to the following system of linear equations.

$$\begin{cases} x - 2y - z + 3t = 1 \\ 2x - 4y + z = 5 \\ x - 2y + 2z - 3t = 4 \end{cases}$$

The given system can be written in the matrix form $AX = B$

$$\begin{pmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

Solution *The augmented matrix is*

$$(A\,|B\,) = \left(\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right)$$

Use the elementary row operations to reduce the augmented matrix $(A\,|B\,)$

Subtracting twice row 1 from row 2 and subtracting row 1 from row 3 gives

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right) \begin{array}{c} R_2 - 2R_1 \longrightarrow R_2 \\ \sim \\ R_3 - R_1 \longrightarrow R_3 \end{array} \left(\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right)$$

Now subtract row 2 from row 3 and multiply row 2 by $\frac{1}{3}$ *to get*

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right) \begin{array}{c} R_3 - R_2 \longrightarrow R_3 \\ \sim \end{array} \left(\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} \frac{1}{3}R_2 \longrightarrow R_2 \\ \sim \end{array} \left(\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

we take it to reduced form by adding row 2 *to row* 1 *:*

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The corresponding reduced system of equations is

$$\begin{aligned} x - 2y + t &= 2 \\ z - 2t &= 1 \\ 0 &= 0 \end{aligned}$$

rank(A) =*rank*$(A\,|B\,) = 2 <$ *number of unknowns,*

Therefore the given system has infinitely many solutions given by

$$\begin{pmatrix} 2+2y-t \\ y \\ 1+2t \\ t \end{pmatrix}$$

Where y, t *is arbitrary in* $\mathbb{R}$.

Chapter 2 Exercise

1. **Exercise 1**

   Find the solution to the following system of linear equations, if any

   (a). $\begin{cases} 6x - 2y + 1 = 0 \\ x - 5y - 3 = 0 \\ 4x + 7y - 17 = 0 \end{cases}$

   (b). $\begin{cases} 8x - 10y - 2 = 0 \\ 2x + y + 6 = 0 \\ 2x + 4y - 20 = 0 \end{cases}$

   (c). $\begin{cases} 2x + 3y - z = 0 \\ 4x + 6y - 3z = 0 \end{cases}$

2. **Exercise 2**

   Do the following systems have a non-zero solution or not? Then find it if it exists (a). $$\begin{cases} 2x + 6y - 4z = 0 \\ x - 8y + 8z = 0 \\ -6x + 9y - 3z = 0 \end{cases},$$

(b). $$\begin{cases} x + 2y - 5z = 0 \\ -6x + 4y - 6z = 0 \\ 12x - 8y + 16z = 0 \end{cases}$$

3. **Exercise 3**

   Find the solution to the following system of linear equations using Gauss's method.

(a). $$\begin{cases} x_1 + 2x_2 + 5x_3 = -9 \\ 2x_1 - 2x_2 + 6x_3 = 4 \\ 3x_1 - 6x_2 - x_3 = 25 \end{cases}$$

(b). $$\begin{cases} 2x_1 - 2x_2 - 4x_3 - 2x_4 = 4 \\ 4x_1 + 2x_2 - 6x_3 + 2x_4 = 12 \\ -x_1 - x_2 - x_3 - x_4 = -7 \end{cases}$$

Chapter 3 Integrals and Primitive functions

Introduction

3.1 Primitive functions

Let f be a continuous function in an interval I, we call F is a primitive function of the function f on the interval I, If F is differentiable at each point of I and the derivative of F is f, in other word:

$$\left(\begin{array}{l}\text{The function } f \text{ is differentiable in an interval } I \\ \text{and } \forall x \in I,\ F'(x) = f(x)\end{array}\right) \Longleftrightarrow \left(\begin{array}{l} F \text{ is a primitive function of } f \\ \text{in an interval } I\end{array}\right)$$

Example 3.1

- The function $F(x) = \sqrt{x}$ is a primitive of $f(x) = \frac{1}{2\sqrt{x}}$ on $]0; +\infty[$,
- The function $F(x) = \arctan x$ is a primitive of $f(x) = \frac{1}{1+x^2}$ on $\mathbb{R}$,
- The function $F(x) = \frac{x^2}{2}$ is a primitive of $f(x) = x$ on $\mathbb{R}$,

  $f(x) = x$ have an infinite number of primitives, such as $\frac{x^2}{2} - 1$, $\frac{x^2}{2} + 4$, $\frac{x^2}{2} - 7$, etc. Thus, all the primitives of x can be obtained by changing the value of c in $F(x) = \frac{x^2}{2} + c$, where c is an arbitrary constant.
- The power function $f(x) = x^n$ has primitive $F(x) = \frac{x^{n+1}}{n+1} + c$ if $n \neq -1$, and $F(x) = \ln|x| + c$ if $n = -1$.

Property *[14, 17]*

- *If f is continuous function on interval I, then f has a primitive function F on I.*
- *Let F be a primitive function of f on I. Then the set of all primitive functions of f on I is:*

$$\{F + c,\ c \in \mathbb{R}\}$$

3.1.1 Primitive functions of elementary functions

By reversing the direction of formulas for derivatives of elementary functions we get the following table of primitive functions [14, 17]:

Definition domain	Function $f(x)$	Primitive $F(x) + C$ (C: constant)
$\mathbb{R}$	x^n	$\frac{x^{n+1}}{n+1} + C$
$]0; +\infty[$	$\frac{1}{x}$	$\ln\|x\| + C$
$\mathbb{R}$	e^x	$e^x + C$
$\mathbb{R}$	$\sin x$	$-\cos x + C$
$\mathbb{R}$	$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + C$
$\mathbb{R}$	$\cos x$	$\sin x + C$
$\mathbb{R}$	$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + C$
$\mathbb{R} - \{(2k+1)\pi/2; k \in \mathbb{Z}\}$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\tan x + C$
$\mathbb{R} - \{(2k+1)\pi/2; k \in \mathbb{Z}\}$	$\tan x$	$-\ln\|\cos x\| + C$
$]-1; 1[$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$]-1; 1[$	$\frac{-1}{\sqrt{1-x^2}}$	$\arccos x + C$
$\mathbb{R}$	$\frac{1}{1+x^2}$	$\arctan x + C$
$\mathbb{R}$	$\frac{-1}{1+x^2}$	$arccotanx + C$

We will also assume knowledge of the following well-known, basic integral formulas:

Function	Primitive function
$f^{'}(x) \cdot f^{n}(x)$	$\frac{1}{n+1} f^{n+1}(x) + C,\, n \neq -1$
$\frac{f^{'}(x)}{f(x)}$	$\ln \lvert f(x) \rvert + C$
$f^{'}(x) \cdot e^{f(x)}$	$e^{f(x)} + C$
$f^{'}(x) \cdot \sin(f(x))$	$-\cos(f(x)) + C$
$f^{'}(x) \cdot \cos(f(x))$	$\sin(f(x)) + C$
$\frac{f^{'}(x)}{\cos^2(f(x))}$	$\tan(f(x)) + C$
$\frac{f^{'}(x)}{\sqrt{1-(f(x))^2}}$	$\arcsin(f(x)) + C$
$\frac{-f^{'}(x)}{\sqrt{1-(f(x))^2}}$	$\arccos(f(x)) + C$
$\frac{f^{'}(x)}{1+(f(x))^2}$	$\arctan(f(x)) + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \arctan \frac{x}{a} + C$

3.2 Types of Integrals

Integration can be classified into two different categories, namely;

- Indefinite Integrals
- Definite Integrals

3.3 Indefinite Integrals

Definition 3.1

*[14, 17] If a function $f(x)$ has one primitive $F(x)$, then it has an **infinite number** of primitives. The set of all primitives $\{F + c,\ c \in \mathbb{R}\}$ of $f(x)$ is called the **indefinite integral** of $f(x)$ with respect to x. The integration of a function f(x) is represented by:*

$$\int f(x)\, dx = F(x) + c \tag{3.1}$$

♣

- The function $f(x)$ under the integral sign is called the integrand.
- The x is the integration variable.

- The symbol dx is the differential of x.
- An arbitrary constant c is said to be a constant of integration.

3.4 Properties of indefinite integrals

[14, 17] Let f and g two continuous functions

1. $\int f^{'}(x)\,dx = f(x) + c,\, c \in \mathbb{R}$
2. $\left(\int f(x)\,dx\right)^{'} = f(x)$
3. ($k =$ constant), $\int k \cdot f(x)dx = k \int f(x)dx$
4. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

Properties 3 and 4 give the linearity of the integral operator in the following equation.

$$\int [\alpha f(x) + \beta g(x)]dx = \alpha \int f(x)dx + \beta \int g(x)dx \tag{3.2}$$

Below some examples are provided to evaluate the indefinite integral using table of usual primitive functions and linearity of the integral.

Example 3.2

1. Evaluate: $\int \left(x^2 + 3x^3 + 7x - 5\right) dx$.

   Using the linearity of the integral in equation (3.2), we have:

$$\begin{aligned}\int \left(x^2 + 3x^3 + 7x - 5\right) dx &= \int x^2dx + \int 3x^3dx + \int 7xdx - \int 5dx \\ &= \int x^2dx + 3\int x^3dx + 7\int xdx - 5\int dx \\ &= \frac{x^3}{3} + 3\frac{x^4}{4} + 7\frac{x^2}{2} - 5x + C\end{aligned}$$

2. Evaluate: $\int \left(7x^3 + 3\cos x\right) dx$.

$$\begin{aligned}\int \left(7x^3 + 3\cos x\right) dx &= \int 7x^3dx + \int 3\cos xdx \\ &= 7\int x^3dx + 3\int \cos xdx \\ &= 7 \cdot \frac{x^4}{4} + 3 \cdot \sin x + C \\ &= \frac{7x^4}{4} + 3\sin x + C\end{aligned}$$

3. Evaluate: $\int(\sin x - 2\cos x)dx$.

$$\int(\sin xdx - 2\cos x)dx = \int \sin xdx - 2\int \cos xdx$$
$$= -\cos x - 2\sin x + C$$

4. Evaluate: $\int \left(3\cos x - \frac{1}{5}e^x\right)dx$.

$$\int \left(3\cos x - \frac{1}{5}e^x\right)dx = 3\int \cos xdx - \frac{1}{5}\int e^x dx$$
$$= 3\sin x - \frac{1}{5}e^x + C$$

3.5 Definite Integrals

Definition 3.2

*[14, 17] Suppose $f(x)$ is a continuous real-valued function on $[a,\ b]$ and also suppose that $F(x)$ is any primitive for $f(x)$. The value $F(b) - F(a)$ is called the **definite integral** of the function f, we read the integral from a to b of the function f and we write:*

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

♣

On a definite integral, above and below the integral symbol are the boundaries of the interval, $[a,\ b]$. The numbers a and b are called the **limits of integration**; where a is the lower limit and b is the upper limit. The definite integral of a real function can be imagined as the area between the x-axis and the curve $y = f(x)$ over an interval $[a,\ b]$, see Fig 3.1

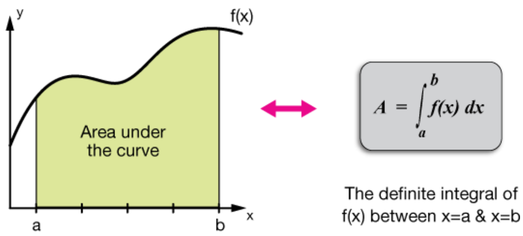


Figure 3.1: Definite Integral (from a to b)

3.6 Properties of definite integrals

The properties of indefinite integrals apply to definite integrals as well, definite integrals also have properties that relate to the limits of integration [14, 17].

1. Multiplication by a constant (k = constant)

$$\int_a^b k \cdot f(x)dx = k \int_a^b f(x)dx$$

2. Reversing the interval Fig 3.2

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

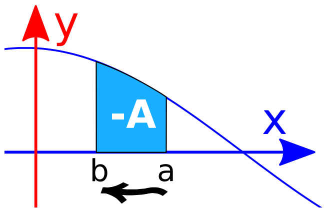


Figure 3.2: Reversing the interval

3. Chasles relation $a < c < b$ Fig 3.3

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

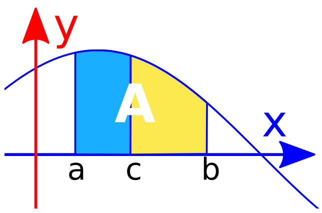


Figure 3.3: Chasles relation for definite integral

4. Interval of zero length Fig 3.4

$$\int_a^a f(x)dx = 0$$

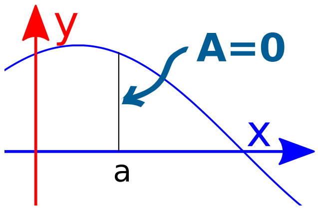


Figure 3.4: Zero integral

5. The integral of a sum functions

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

6. Let $a \leq b$

   - If f is a positive function on $[a,\ b]$; then $\int_a^b f(x)dx \geq 0$
   - If f is a negative function on $[a,\ b]$; then $\int_a^b f(x)dx \leq 0$
   - [If $\forall x \in [a,\ b]\ ,\ f\ (x) \leq g\ (x)] \Longrightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

7. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx$

8. If f is an even function, then

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

9. if f is an odd function, then

$$\int_{-a}^a f(x)dx = 0$$

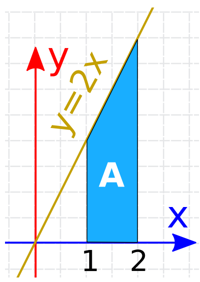
10. If f is periodic, of period T then

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$$

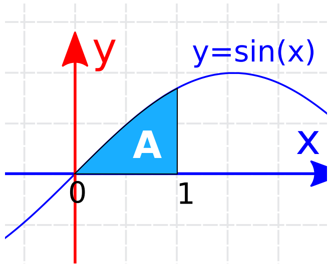
Example 3.3

1. Determine the value of $\int_1^2 2xdx$

   $\int_1^2 2xdx = [x^2]_1^2 = 2^2 - 1^2 = \mathbf{3}$



2. Determine the value of $\int_0^1 \sin(x)dx$



$$\begin{aligned}\int_0^1 \sin x dx &= [-\cos x]_0^1 = -\cos 1 - (-\cos 0) \\ &= 0.46\end{aligned}$$

3. Evaluate: $\int_1^5 |x-3|\, dx$

$$|x-3| = \begin{cases} x-3, \ x \geq 3 \\ -x+3, \ x < 3 \end{cases}$$

$$\begin{aligned}\int_1^5 |x-3|\, dx &= \int_1^3 (-x+3)\, dx + \int_3^5 (x-3)\, dx \\ &= \left[\frac{-x}{2} + 3x\right]_1^3 + \left[\frac{x}{2} - 3x\right]_3^5 \\ &= \frac{5}{2} - 9 = \frac{-13}{2}\end{aligned}$$

3.7 Integration involves Trigonometric, Logarithmic and Exponential Functions

There are many methods of integration involves Trigonometric, Logarithmic and Exponential Functions, that are used to solve mathematical operations.

3.7.1 Integration of Trigonometric Functions

3.7.1.1 Integral of the types

- $\int f(\sin x) \cos x dx$

- $\int f(\cos x)\sin x dx$
- $\int f((\tan x) dx$

1. If the integral is in the form $\int f(\sin x)\cos x dx$

   We use the following variable change: $u = \sin x, \quad du = \cos x dx$

   The integral is written: $\int f(u) du$.

2. If the integral is in the form $\int f(\cos x)\sin x dx$

   We use the following variable change: $u = \cos x, \quad du = \sin x dx$

   The integral is written : $\int f(u) du$

3. If the integral to integrate only depends on $\tan x$: $\int f((\tan x) dx$

   Let's use the following variable change:

   $u = \tan x, \quad x = \arctan u \Rightarrow dx = \frac{du}{u^2+1}$

   We obtain $\int f(\tan x) dx = \int f(u)\frac{du}{u^2+1}$

   Example 3.4

   Find $\int \sin^5 x \cos x dx$

   By changing the variable:$u = \sin x, \quad du = \cos x dx$

$$\begin{aligned}\int \sin^5 x \cos x dx &= \int u^5 du \\ &= \frac{1}{6}u^6 + c \\ \int \sin^5 x \cos x dx &= \frac{1}{6}\sin^6 x + c\end{aligned}$$

   Example 3.5

   Find $\int \cos^2 x \sin x dx$

   By changing the variable: $u = \cos x, \quad du = -\sin x dx$

$$\begin{aligned}\int \cos x^3 \sin x dx &= \int -u^3 du \\ &= -\frac{1}{4}u^4 + c \\ \int \cos x^3 \sin x dx &= -\frac{1}{4}\cos x^4 + c\end{aligned}$$

   Example 3.6

   Find $\int \sin^3 x dx$

$\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int (1 - \cos x^2) \sin x dx$

By changing the variable: $u = \cos x, \quad du = -\sin x dx$

$$\begin{aligned} \int \left(1 - \cos^2 x\right) \sin x dx &= \int -\left(1 - u^2\right) du \\ &= \int u^2 du - \int du \\ &= \frac{1}{3} u^3 - u + c \\ \int \sin^3 x dx &= \frac{1}{3} \cos^3 x - \cos x + c \end{aligned}$$

Example 3.7

$\int \frac{\tan x}{\cos^2 x} dx$

We have: $\cos^2 x = \frac{1}{1+\tan^2 x} \Longrightarrow \frac{1}{\cos^2 x} = 1 + \tan^2 x.$

By substituting this relation into the above integral:

$$\begin{aligned} \int \frac{\tan x}{\cos^2 x} dx &= \int \tan x \left(1 + \tan^2 x\right) dx \\ &= \int \tan x dx + \int \tan^3 x dx \\ &= \int t \frac{dt}{t^2+1} + \int t^3 \frac{dt}{t^2+1} \end{aligned}$$

Were $t = \tan x$, and $\left(\frac{t^3}{t^2+1} = t - \frac{t}{t^2+1}\right)$

$$\begin{aligned} \int \frac{\tan x}{\cos^2 x} dx &= \int t \frac{dt}{t^2+1} + \int \left(t - \frac{t}{t^2+1}\right) dt \\ &= \int t \frac{dt}{t^2+1} + \int t dt - \int t \frac{dt}{t^2+1} \\ &= \int t dt = \frac{1}{2} t^2 + c \end{aligned}$$

Substituting the value of t, we get,

$$\int \frac{\tan x}{\cos^2 x} dx = \frac{1}{2} \tan^2 x + C.$$

3.7.1.2 Integral of the types:

- $\int \sin px \cos qx dx,$
- $\int \sin px \sin qx dx$
- $\int \cos px \cos qx dx$

In this case, we use the following formulas:

$$\sin px \cos qx = \frac{1}{2}[\sin(p+q)x + \sin(p-q)x]$$
$$\sin px \sin qx = \frac{1}{2}[\cos(p-q)x - \cos(p+q)x]$$
$$\cos px \cos qx = \frac{1}{2}[\cos(p-q)x + \cos(p+q)x]$$

Example 3.8

Find $\int \cos\frac{x}{4}\cos\frac{x}{3}dx$

$$\begin{aligned}
\cos\frac{x}{4}\cos\frac{x}{3}dx &= \frac{1}{2}\left[\cos\left(\frac{1}{4}-\frac{1}{3}\right)x + \cos\left(\frac{1}{4}+\frac{1}{3}\right)x\right] \\
&= \frac{1}{2}\left[\cos\left(\frac{-1}{12}\right)x + \cos\left(\frac{7}{12}\right)x\right] \\
\int \cos\frac{x}{4}\cos\frac{x}{3}dx &= \int \frac{1}{2}\left[\cos\frac{-1}{12}x + \cos\frac{7}{12}x\right]dx \\
&= \frac{1}{2}\left[\int \cos\frac{x}{12}dx + \int \cos\frac{7}{12}xdx\right] \\
&= \frac{1}{2}\left[12\sin\frac{1}{12}x + \frac{12}{7}\sin\frac{7}{12}x\right] + c
\end{aligned}$$

$$\int \cos\frac{x}{4}\cos\frac{x}{3}dx = 6\sin\frac{x}{6} + \frac{6}{7}\sin\frac{7}{12}x + c$$

3.7.1.3 Integral of the types:

$$I_{p,q} = \int \sin^p x \cos^q x dx \tag{3.3}$$

we consider two cases:

1. **First case** $p = 2k+1$ is an odd number

$$\begin{aligned}
I_{p,q} &= \int \sin^{(2k+1)} x \cos^q x dx \\
&= \int \sin^{2k} x \sin x \cos^q x dx = \int \left(1-\cos^2 x\right)^k \cos^q x \sin x dx
\end{aligned}$$

In this case we use the following variable change

$$u = \cos x \quad du = -\sin x dx$$

Substituting these expressions into equation 3.3, we find

$$I_{p,q} = \int \left(1 - \cos^2 x\right)^k \cos^q x \sin x dx = -\int \left(1 - u^2\right)^k u^q du$$

We proceed in the same way if q is an odd positive number

2. **Second case** if p and q two even positive numbers

   Using the following formulas, we can transform the expression 3.3:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \;\; \cos^2 x = \frac{1}{2}(1 + \cos 2x) \text{ and } \sin x \cos x = \frac{1}{2}\sin 2x$$

Example 3.9

Find $\int \sin^3 x \cos^4 x dx$

$p = 3$ is an odd positive number

$$\begin{aligned}\int \sin^3 x \cos^4 x dx &= \int \sin^2 x \sin x \cos^4 x dx \\ &= \int \left(1 - \cos^2 x\right) \sin x \cos^4 x dx \\ &= \int \cos^4 x \sin x dx - \int \cos^6 x \sin x dx\end{aligned}$$

With the change of the variable mentioned above, we obtain.:

$$\int \sin^3 x \cos^4 x dx = \int -u^4 du + \int u^6 du = -\frac{1}{7}u^7 + \frac{1}{7}u^7 + c$$

$$\int \sin^3 x \cos^6 x dx = -\frac{1}{7}\cos^7 x + \frac{1}{7}\cos^7 x + c$$

3.7.1.4 integral of type

$I = \int R\left(\sin x, \;\cos x\right) dx$

where R is rational fraction of the functions $\sin$, $\cos$

In general method, we put $t = \tan\frac{x}{2}$, thus, we find the following relationships

$t = \tan\frac{x}{2} \Longrightarrow x = 2\arctan t \Longrightarrow dx = \frac{2}{1+t^2}dt$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

thus, we find

$$I = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt$$

Remark

- In integrals of the form $I = \int R(\sin x) \cdot \cos x dx$, we put $t = \sin x$
- In integrals of the form $I = \int R(\cos x) \cdot \sin x dx$, we put $t = \cos x$
- In integrals of the form $I = \int R(\tan x)\, dx$, we put $t = \tan x$

Example 3.10

Find $\int \frac{\sin x}{1+\cos x}dx$

Let's put $t = \tan\frac{x}{2} \Longrightarrow dx = \frac{2}{1+t^2}dt$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$$\begin{aligned}
\int \frac{\sin x}{1+\cos x}dx &= \int \frac{2t}{(1+t^2)\left(1+\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}dt \\
&= \int \frac{2t}{1+t^2}dt \\
&= \ln\left|1+t^2\right| + C \\
&= \ln\left|1+\tan^2\frac{x}{2}\right| + C
\end{aligned}$$

3.7.2 Integration by Substitution

In order to solve the difficulty of some integration, we use the substitution by introducing a new independent variable $t = g(x)$ in the integral function $\int f(t)dt$, we get, $\frac{dt}{dx} = g^{'}(x)$ or $dt = g^{'}(x)\, dx$ Thus, from the above substitution ,we get,

$$\boxed{\int f(g(x)).g^{'}(x)\, dx = \int f(t).dt}$$

The substitution rule can transform a complicated integral into a simple one.

Remark

- The substitution Rule for definite integrals is given by

$$\text{If } t = g(x), \text{ then } \int_a^b f(g(x))g^{'}(x)dx = \int_{g(a)}^{g(b)} f(t)dt$$

Example 3.11

Evaluate the following integrals

1. $I_1 = \int x \sin\left(x^2+3\right) dx$
2. $I_2 = \int \frac{1}{x \ln x} dx$
3. $I_3 = \int \frac{e^{\arctan x}}{1+x^2} dx$
4. $I_4 = \int \frac{\tan^2 x - 2\tan x + 5}{\cos^2 x} dx$
5. $I_5 = \int x\sqrt{x+1} dx$
6. $I_6 = \int_2^3 \frac{1}{(\ln x)x} dx$

Solution

- $I_1 = \int x \sin\left(x^2+3\right) dx$

  Let $t = x^2 + 3$, therefore $dt = 2xdx$.

  $dt = 2xdx \quad \Rightarrow \quad \frac{1}{2}dt = xdx.$

  We can now substitute

$$\begin{aligned} \int x \sin\left(x^2+3\right) dx &= \int \sin \underbrace{\left(x^2+3\right)}_{t} \underbrace{xdx}_{\frac{1}{2}dt} \\ &= \int \frac{1}{2} \sin t dt \\ &= -\frac{1}{2}\cos t + C \quad \text{(by replacing } t \text{ with } x^2+3 \text{ we get)} \\ &= -\frac{1}{2}\cos\left(x^2+3\right) + C. \end{aligned}$$

  Thus $\int x \sin\left(x^2+3\right) dx = -\frac{1}{2}\cos\left(x^2+3\right) + C$.

- $I_2 = \int \frac{1}{x \ln x} dx$

  We choose $t = \ln x$ then $dt = 1/xdx$, which gives

$$\begin{aligned} \int \frac{1}{x \ln x} dx = \int \underbrace{\frac{1}{\ln x}}_{1/t} \underbrace{\frac{1}{x} dx}_{dt} \\ = \int \frac{1}{t} dt \\ = \ln|t| + C \\ = \ln|\ln x| + C. \end{aligned}$$

- $I_3 = \int \frac{e^{\arcsin x}}{\sqrt{1-x^2}} dx$

Solution:

Let $t = \arcsin x$, then $dt = \frac{1}{\sqrt{1-x^2}}dx$

$$\begin{aligned} I_3 &= \int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}dx = \int e^t dt \\ &= e^t + C \end{aligned}$$

Substituting the value of t, we get

$$I_3 = \int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}dx = e^{\arcsin x} + C$$

- $I_4 = \int \frac{\tan^2 x - 2\tan x + 5}{\cos^2 x}dx$

Let $t = \tan x$ makes $dt = \frac{1}{\cos^2 x}dx$

$$\begin{aligned} I_4 &= \int \frac{\tan^2 x - 2\tan x + 5}{\cos^2 x}dx = \int \left(t^2 - 2t + 5\right) dt \\ &= \frac{t^3}{3} - t^2 + 5t + C \end{aligned}$$

Substituting the value of t, we get

$$I_4 = \int \frac{\tan^2 x - 2\tan x + 5}{\cos^2 x}dx = \frac{1}{3}\tan^3 x - \tan^2 x + 5\tan x + C$$

$I_5 = \int x\sqrt{x+1}dx.$

Solution

Put $t = x + 1$, then $\frac{dt}{dx} = 1$, so $dt = dx$, then

$$\int x\sqrt{x+1}dx = \int x(x+1)^{\frac{1}{2}}dx = \int xt^{\frac{1}{2}}du.$$

By the substitution $t = x + 1$, it follows that $x = t - 1$, then we obtain

$$\begin{aligned} \int xt^{1/2}dt &= \int (t-1)t^{\frac{1}{2}}dt \\ &= \int \left(t \cdot t^{\frac{1}{2}} - t^{\frac{1}{2}}\right) dt \\ &= \int \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right) dt \end{aligned}$$

Thus

$$\begin{aligned} I_5 &= \frac{1}{\frac{3}{2}+1}t^{\frac{3}{2}+1} + \frac{1}{1/2+1}t^{\frac{1}{2}+1} + C \\ &= \frac{2}{5}t^{5/2} + \frac{2}{3}t^{3/2} + C \\ &= \frac{2}{5}\sqrt{t}^5 + \frac{2}{3}\sqrt{t}^3 + C \\ &= \frac{2}{5}\sqrt{x+1}5 + \frac{2}{3}\sqrt{x+1}^3 + C \end{aligned}$$

3.7.3 Integration by Parts

Integration by Parts can be used to integrate any given function if the integration function is represented as a multiple of two or more functions.

The product rule of derivation will be the starting point for the integration by part:

$$(f(x).g(x))^{'} = f(x).g^{'}(x) + f^{'}(x).g(x)$$

Now, integrate both sides of this.

$$\begin{aligned} \int (f(x).g(x))^{'}\,dx &= \int \Big(f(x).g^{'}(x) + f^{'}(x).g(x)\Big)\,dx \\ \Rightarrow f(x).g(x) &= \int f(x).g^{'}(x)dx + \int f^{'}(x).g(x)dx \end{aligned}$$

The integration by parts formula can be reached by rewriting the formula as follows:

$$\boxed{\int f(x).g^{'}(x)dx = f(x).g(x) - \int f^{'}(x).g(x)dx}$$

Remark

- In definite integral, we use the formula:

$$\int_a^b f(x).g^{'}(x)dx = [f(x).g(x)]_a^b - \int_a^b f^{'}(x).g(x)dx$$

- To use this formula effectively, it is necessary to accurately identify both the $f(x)$ and the $g^{'}(x)$. The choice is not randomly made.
- Integration by parts can be used more than once.

Example 3.12

Evaluate the following integrals

1. $I_1 = \int xe^x dx$
2. $I_2 = \int x \cos x dx$
3. $I_3 = \int x^n \ln x dx$
4. $I_4 = \int \arcsin x dx$
5. $I_5 = \int x \arctan x dx$
6. $I_6 = \int e^x \sin x dx$
7. $I_7 = \int x^2 e^{-3x} dx$

Solutions

- $I_1 = \int xe^x dx$

  We will integrate this by parts, using the formula

$$\int f'g = fg - \int fg'$$

  Let $g(x) = x$ and $f'(x) = e^x$ Then we obtain g' and f by differentiation and integration.

$$\begin{cases} f'(x) = e^x \\ g(x) = x \end{cases} \Longrightarrow \begin{cases} f(x) = e^x \\ g'(x) = 1 \end{cases}$$

  $\int f'g = fg - \int fg'$ becomes

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

- $I_2 = \int x \cos x dx$

  Let $g(x) = x$ and $f'(x) = \cos x$

  Then we obtain g' and f by differentiation and integration.

$$\begin{cases} f'(x) = \cos x \\ g(x) = x \end{cases} \Longrightarrow \begin{cases} f(x) = \sin x \\ g'(x) = 1 \end{cases}$$

  $\int f'g = fg - \int fg'$ becomes

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x - (-\cos x) = x \sin x + \cos x + C$$

- $I_3 = \int x^n \ln x dx$

  Let $g(x) = \ln x$ and $f'(x) = x^n$

Then we obtain g' and f by differentiation and integration.

$$\begin{cases} f'(x) = x^n \\ g(x) = \ln x \end{cases} \Longrightarrow \begin{cases} f(x) = \frac{x^{n+1}}{n+1} \\ g'(x) = \frac{1}{x} \end{cases}$$

$\int f'g = fg - \int fg'$ becomes

$$\begin{aligned} I_3 &= \int x^n \ln x dx = \ln x \cdot \frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \\ &= \frac{x^{n+1} \ln x}{n+1} - \frac{1}{n+1} \int x^n dx \\ &= \frac{x^{n+1} \ln x}{n+1} - \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} + C \\ &= \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C \end{aligned}$$

- $I_4 = \int \arcsin x dx$

  Let $g(x) = \arcsin x$ and $f'(x) = 1$.

  Then we obtain g' and f by differentiation and integration.

$$\begin{cases} f'(x) = 1 \\ g(x) = \arcsin x \end{cases} \Longrightarrow \begin{cases} f(x) = x \\ g'(x) = \frac{1}{\sqrt{1-x^2}} \end{cases}$$

  We compute the integral $\int \frac{x}{\sqrt{1-x^2}} dx$ by substitution.

  Let $t = 1 - x^2$. Then $dt = -2xdx$ and so $dx = \frac{dt}{-2x}$.

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} dx &= \int \frac{x}{\sqrt{t}} \frac{dt}{-2x} \\ &= -\frac{1}{2} \int \frac{1}{\sqrt{t}} dt \\ &= -\frac{1}{2} \int t^{-1/2} du \\ &= -\frac{1}{2} \frac{t^{1/2}}{\frac{1}{2}} + C \\ &= -\sqrt{t} + C \\ &= -\sqrt{1-x^2} + C \end{aligned}$$

  Thus the entire integral is

$$I_4 = \int \arcsin x dx = x \arcsin x - \left(-\sqrt{1-x^2} \right) + C = x \arcsin x + \sqrt{1-x^2} + C$$

- $I_5 = \int x \arctan x dx$

Let $g(x) = \arctan x$ and $f'(x) = x$ Then we obtain g' and f by differentiation and integration.

$$\begin{cases} f'(x) = x \\ g(x) = \arctan x \end{cases} \Longrightarrow \begin{cases} f(x) = \frac{x^2}{2} \\ g'(x) = \frac{1}{x^2+1} \end{cases}$$

$\int f'g = fg - \int fg'$ becomes

$$\begin{aligned} I_5 &= \int x \arctan x dx = \frac{x^2}{2} \arctan x - \int \frac{x^2}{2} \cdot \frac{1}{x^2+1} dx \\ &= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{x^2+1} dx \\ &= \frac{x^2}{2} \arctan x - \frac{1}{2} \left[\int \frac{x^2+1}{x^2+1} dx - \int \frac{1}{x^2+1} dx \right] \\ &= \frac{x^2}{2} \arctan x - \frac{1}{2} [x - \arctan x] + C \end{aligned}$$

- $I_6 = \int e^x \sin x dx$

Let $g(x) = \sin x$ and $f'(x) = e^x$ (Notice that if yo choose, $g(x) = e^x$ and $f'(x) = \sin x$ would also work.) We obtain g' and f by differentiation and integration.

$$\begin{cases} f'(x) = e^x \\ g(x) = \sin x \end{cases} \Longrightarrow \begin{cases} f(x) = e^x \\ g'(x) = \cos x \end{cases}$$

$\int f'g = fg - \int fg'$ becomes

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

It looks like our method produced a new integral, $\int e^x \cos x dx$ that also requires integration by parts.

Let $g(x) = \cos x$ and $f'(x) = e^x$. We obtain g' and f by differentiation and integration.

$$\begin{cases} f'(x) = e^x \\ g(x) = \cos x \end{cases} \Longrightarrow \begin{cases} f(x) = e^x \\ g'(x) = -\sin x \end{cases}$$

$\int f'g = fg - \int fg'$ becomes

$$\int e^x \cos x dx = e^x \cos x - \int e^x (-\sin x) dx = e^x \cos x + \int e^x \sin x dx$$

Thus $\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$

Now the result contains the original integral, $\int e^x \sin x$.

Recall that we denote $\int e^x \sin x$ by I_6. Let us review the computation again:

$$\begin{aligned}\int e^x \sin x dx &= e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x dx\right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x dx\end{aligned}$$

This is the same as

$$I_6 = e^x \sin x - e^x \cos x - I_6$$

This is an equation that we can solve for I_6.

$$2I_6 = e^x \sin x - e^x \cos x I_6 = \frac{1}{2}e^x(\sin x - \cos x)$$

Thus the answer is

$$I_6 = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

- $I_7 = \int x^2 e^{-3x} dx$

We will need to integrate by parts twice. First, let $f'(x) = e^{-3x}$ and $g(x) = x^2$. Then

$$\begin{cases} f'(x) = e^{-3x} \\ g(x) = x^2 \end{cases} \Longrightarrow \begin{cases} f(x) = -\frac{1}{3}e^{-3x} \\ g'(x) = 2x \end{cases}$$

$\int f'g = fg - \int fg'$ becomes

$$\int x^2 e^{-3x} dx = -\frac{1}{3}e^{-3x}\left(x^2\right) - \int \left(-\frac{1}{3}e^{-3x}\right) 2x dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int xe^{-3x} dx$$

and we can compute $\int xe^{-3x} dx$ by integrating by parts. Let $f'(x) = e^{-3x}$ and $g(x) = x$. Then

$$\begin{cases} f'(x) = e^{-3x} \\ g(x) = x \end{cases} \Longrightarrow \begin{cases} f(x) = -\frac{1}{3}e^{-3x} \\ g'(x) = 1 \end{cases}$$

$\int f'g = fg - \int fg'$ becomes

$$\begin{aligned}\int xe^{-3x}dx &= -\frac{1}{3}e^{-3x}(x) - \int\left(-\frac{1}{3}e^{-3x}\right)dx \\ &= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx \\ &= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{-3x}\right) + C \\ &= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C\end{aligned}$$

We need to compute the integral $\int x^2e^{-3x}dx$. So far we had this much:

$$\int x^2e^{-3x}dx = -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}dx$$

To this we substitute our result $\int xe^{-3x}dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$:

$$\begin{aligned}\int x^2e^{-3x}dx &= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}dx \\ &= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C_1\right) \\ &= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C\end{aligned}$$

3.8 Integration of rational functions

[17] Understanding the meaning of partial fractions and how to write them is crucial before studying the integration process using them.

We know that a rational function is a ratio of two polynomials $P(x)/Q(x)$. Now, if the degree of $P(x)$ is lesser than the degree of $Q(x)$, then it is a **proper fraction**, else it is an **improper fraction**.

1. **First case** if $P(x)/Q(x)$ is an improper fraction

   In this case $P(x)/Q(x) = H(x) + P1(x)/Q(x)$, where $H(x)$ is a polynomial and $P1(x)/Q(x)$ is a proper rational fraction.

2. **Second case** if $P(x)/Q(x)$ is a proper fraction

   Let's say that we want to evaluate $\int[P(x)/Q(x)]dx$, where $P(x)/Q(x)$ is a proper rational fraction.

   In this case, it is possible to write the integrand as a sum of simpler rational functions by using partial fraction decomposition.

We follow the following rules in the decomposition process:

- Each linear factor $(x-r)$ of $Q(x)$ corresponds to a partial fraction $\frac{A}{x-r}$
- Each power linear factor $(x-r)^m$ of $Q(x)$ has a corresponding m partial fractions

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}$$

where A_i are constants

- Each irreducible quadratic factor ax^2+bx+c, $(ax^2+bx+c$ has no real roots; $\Delta < 0)$ of $Q(x)$ corresponds to a partial fraction $\frac{Bx+C}{(ax^2+bx+c)}$, where B and C are constants.
- Each irreducible power quadratic factor $(ax^2+bx+c)^n$ of $Q(x)$ has a corresponding n partial fractions

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \frac{B_3x+C_3}{(ax^2+bx+c)^3} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

where B_i and C_i are constants.

To find the constants, we equate $\frac{P(x)}{Q(x)}$ with the sum of all these partial fractions, then we solve the undetermined coefficients.

After the decomposition process, we discuss the integration of partial fractions, which takes one of the following forms:

1. $\int \frac{1}{(x-r)^n} dx$
2. $\int \frac{1}{(t^2+1)^n} dt$
3. $\int \frac{Ax+B}{(ax^2+bx+c)^n} dx$

Integration of partial fractions

1. $I = \int \frac{1}{(x-r)^n} dx$

   There are two cases

   - $n = 1 \longrightarrow I = \ln|x-r| + C$
   - $n \neq 1 \longrightarrow I = \frac{-1}{(n-1)(n-r)^{n-1}} + C$

2. $I_n = \int \frac{1}{(t^2+1)^n} dt$

   There are three cases

- $n = 1 \longrightarrow I_n = \arctan x + C$
- $n = \frac{1}{2} \longrightarrow I_n = \ln\left|t + \sqrt{t^2+1}\right| + C$
- $n \neq 1, n \neq \frac{1}{2} \longrightarrow$We use the Substitution variable $t = \tan x$

3. $I_n = \int \frac{Ax+B}{(ax^2+bx+c)^n} dx$

   When $ax^2 + bx + c$ is irreducible quadratic factor $(\Delta < 0)$

   We integrate this I_n through the following steps

   - **First step:** we write the numerator in terms of the derivative of the denominator $(ax^2 + bx + c)$ i.e

   $$\frac{Ax+B}{(ax^2+bx+c)^n} = \frac{\alpha\,(2ax+b)}{(ax^2+bx+c)^n} + \frac{\beta}{(ax^2+bx+c)^n}$$

   - **Second step** we use the typical form $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$ for the fraction $\frac{\beta}{(ax^2+bx+c)^n}$
   - **Third step** we use the Substitution variable $t = \sqrt{\frac{4a^2}{-a}}\left(x + \frac{b}{2a}\right)$

   We obtain

   $$I_n = \alpha \int \frac{(2ax+b)}{(ax^2+bx+c)^n} dx + \lambda \int \frac{1}{(t^2+1)^n} dt$$

   The integral $\int \frac{(2ax+b)}{(ax^2+bx+c)^n} dx$ in the forme $\int \frac{f'(x)}{f(x)} dx$

   The integral $\int \frac{1}{(t^2+1)^n} dt$ in the forme 2.

Example 3.13

Integrate the following integral by the method of partial fractions.

$\int \frac{3x+11}{x^2-x-6} dx$

First, we factor the denominator as much as possible and then, we obtain the form of the partial fraction decomposition.

$$\frac{3x+11}{x^2-x-6} = \frac{A_1}{x-3} + \frac{A_2}{x+2} \tag{3.4}$$

Then,

$$\frac{3x+11}{(x+2)(x-3)} = \frac{A_1(x+2) + A_2(x-3)}{(x+2)(x-3)}$$

Now, we need to found A_1 and A_2 , the numerators of these two are equal for every x, so $3x + 11 =$

$A_1(x+2) + A_2(x-3)$

Two ways to proceed are available at this point. The first option is always successful, but it's a lot more work. The second option is often quicker when it does work. We'll use the quickest method in this instance since both will work, but we'll examine the other method in future examples.

- To find A_1, we multiply both sides of the equality 3.4 by $(x-3)$,

$$\frac{3x+11}{(x+2)} = A_1 + \frac{A_2\,(x-3)}{x+2}$$

Then we replace x with 3, we found $A_1 = 4$

- To find A_2, we multiply both sides of the equality 3.4 by $(x+2)$,

$$\frac{3x+11}{(x-3)} = \frac{A_1\,(x+2)}{x-3} + A_2$$

Then we replace x with -2, we found $A_2 = \frac{5}{-5} = -1$, so

$$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} + \frac{-1}{x+2}$$

Now, we can integrate

$$\begin{aligned}\int \frac{3x+11}{x^2-x-6}dx &= \int \left(\frac{4}{x-3} - \frac{1}{x+2}\right) dx \\ &= 4\int \frac{1}{x-3}dx - \int \frac{1}{x+2}dx \\ &= 4\ln|x-3| - \ln|x+2| + C\end{aligned}$$

Example 3.14

Integrate the following integral by the method of partial fractions.

$$\int \frac{x^2-29x+5}{(x-4)^2\,(x^2+3)}dx$$

The partial fraction decomposition is

$$\begin{aligned}\frac{x^2-29x+5}{(x^2-4)\,(x^2+3)} &= \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3} \\ \frac{x^2-29x+5}{(x^2-4)\,(x^2+3)} &= \frac{A\,(x-4)\,(x^2+3) + B\,(x^2+3) + (Cx+D)\,(x-4)^2}{(x-4)^2\,(x^2+3)}\end{aligned}$$

Then we obtain,

$$x^2-29x+5 = A\,(x-4)\left(x^2+3\right) + B\left(x^2+3\right) + (Cx+D)\,(x-4)^2$$

We propagate the right side and collect all the like terms together, then we get:

$$x^2 - 29x + 5 = (A + C)x^3 + (-4A + B - 8C + D)x^2 + (3A + 16C - 8D)x - 12A + 3B + 16D$$

To find A, B, C, and D, we must establish the coefficients of like powers of x equal

$$\left.\begin{array}{lll}\text{Coefficient of } x^3: & A + C & = 0 \\ \text{Coefficient of } x^2: & -4A + B - 8C + D & = 1 \\ \text{Coefficient of } x^1: & 3A + 16C - 8D & = -29 \\ \text{The constants:} & -12A + 3B + 16D & = 5\end{array}\right\} \Rightarrow \begin{array}{c} A = 1 \\ B = -5 \\ C = -1 \\ D = 2 \end{array}$$

Now, we calculate the above integral.

$$\begin{aligned}\int \frac{x^2 - 29x + 5}{(x-4)^2 (x^2+3)} dx &= \int \left(\frac{1}{x-4} - \frac{5}{(x-4)^2} + \frac{-x+2}{x^2+3} \right) dx \\ &= \int \left(\frac{1}{x-4} - \frac{5}{(x-4)^2} - \frac{x}{x^2+3} + \frac{2}{x^2+3} \right) dx \\ &= \int \frac{1}{x-4} dx - \int \frac{5}{(x-4)^2} dx - \int \frac{x}{x^2+3} dx \\ &\quad + \int \frac{2}{x^2+3} dx \\ &= \ln|x-4| + \frac{5}{x-4} - \frac{1}{2} \ln\left|x^2+3\right| \\ &\quad + \frac{2}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C\end{aligned}$$

Example 3.15

Find $\int \frac{4x+5}{x^2+x+2} dx$

Solution

$Q(x) = x^2 + x + 2 \Longrightarrow \Delta = -7 < 0$

First step: we write the numerator in terms of the derivative of the denominator

$$\begin{aligned}\frac{4x+5}{x^2+x+2} &= \frac{\alpha(2x+1)}{x^2+x+2} + \frac{\beta}{x^2+x+2} \\ &= \frac{\alpha(2x+1) + \beta}{x^2+x+2}\end{aligned}$$

$\alpha(2x+1)+\beta = 4x+5$ so,

$$\begin{cases} 2\alpha = 4 \\ \alpha+\beta = 5 \end{cases} \Longrightarrow \begin{cases} \alpha = 2 \\ \beta = 3 \end{cases}$$

$$\frac{4x+5}{x^2+x+2} = \frac{2(2x+1)}{x^2+x+2} + \frac{3}{x^2+x+2}$$

$$\begin{aligned}\int \frac{4x+5}{x^2+x+2} &= 2\int \frac{(2x+1)}{x^2+x+2}dx + 3\int \frac{1}{x^2+x+2}dx \\ &= 2\ln\left|x^2+x+2\right| + 3J + C\end{aligned}$$

Second step: to find the integral J we use the typical form

$$x^2+x+2 = \left(x+\frac{1}{2}\right)^2 + \frac{7}{4}$$

Third step: we use the substitution variable

$$t = \sqrt{\frac{4}{7}}\left(x+\frac{1}{2}\right), \text{then} dt = \sqrt{\frac{4}{7}}dx$$

$$\begin{aligned}J &= \int \frac{1}{x^2+x+2}dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx \\ &= \int \frac{\sqrt{\frac{7}{4}}}{\frac{7}{4}\left[\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right)^2+1\right]}dx \\ &= \frac{\sqrt{\frac{7}{4}}}{\frac{7}{4}}\int \frac{dt}{t^2+1} \\ &= 3\sqrt{\frac{4}{7}}\arctan\left(\sqrt{\frac{4}{7}}\left(x+\frac{1}{2}\right)\right) + C\end{aligned}$$

Chapter 3 Exercise

1. **Exercise 1**

   Calculate the following integrals:

$\int_{-2}^{+2} \frac{dx}{x^7}$, $\int e^{-5x} dx$, $\int \cos x \sin^5 x dx$

$\int \frac{2x^2}{x^3+1} dx$, $\int_0^{\frac{\pi}{6}} \sqrt{\sin x} \cos x dx$, $\int_1^3 \frac{\sqrt{x-1}}{x} dx$

$\int x\left(x^2+1\right)^{2024} dx$, $\int \frac{x^2}{\sqrt{1-x^3}} dx$, $\int \frac{4x}{(x^2-5)^{1/7}} dx$

$\int_{-3}^{3} \sin^9 x dx$, $\int_0^{\pi/3} \left(\cos^2 x - \sin^2 x\right) dx$, $\int \frac{\cos(\tan x)}{\cos^2 x} dx$

$\int x \sin x dx$, $\int e^x \sin ax dx$, $\int x^2 \ln x dx$

$\int x^2 e^{ax} dx$, $\int \frac{x^2}{\sqrt{5+x^3}} dx$, $\int \frac{3x+1}{3x-1} dx$

$\int \frac{\sqrt{arctgx}}{x^2+1} dx$, $\int_0^1 \ln(x+1) dx$, $\int \frac{\sin x}{\cos^3 x} dx$

$\int_0^{\frac{\pi}{4}} \tan x dx$, $\int \frac{1}{x^4-x} dx$, $\int \frac{1}{x(1-x^2)} dx$

$\int_0^{2\pi} \sin^5(3x)\cos(3x) dx$, $\int_1^4 \frac{1}{(3x-7)^2} dx$, $\int_0^1 \frac{dx}{(x^2+1)^2}$

Chapter 4 Differential Equations

Introduction

- ❑ *Classification of Differential Equations*
- ❑ *Solution of Differential equations*
- ❑ *First Order Differential Equations*
  - *Linear Equation*
    - *Linear homogeneous equation*
    - *Linear non-homogeneous*
  - *Nonlinear Equation*
    - *Bernouli equation*
    - *Separable equations*
    - *Euler Homogeneous*
- ❑ *Second Order Differential Equations*
  - *Linear homogeneous equation*
  - *Linear non-homogeneous*

In this chapter we study several types of differential equations and their corresponding methods of solution.

Definition 4.1 (Ordinary differential equations)

An ordinary differential equation is an equation relating an unknown function y depends on a single independent variable x over an interval I, and contains one or several its derivatives $y^{'}, y^{''}, ..., y^{(n)}$, it can be written in the form:

$$F\left[x, y, y^{'}, y^{''}, ..., y^{(n)}\right] = 0 \tag{4.1}$$

♣

In this course, we will only focus on ordinary differential equations, so we won't use the term ordinary anymore.

Example 4.1

We list the following differential equations:

$$\left(y^{''}\right)^2 + xy^{'}y - \sin x = 0$$

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} + 5y = \cos t$$

$$\begin{aligned}\sqrt{x}\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}y &= 0\\ 3y^{''}+\cos xy^{'}+y &= x^2\end{aligned}$$

4.1 Classification of Differential Equations

Differential equations can be classified into various categories based on their properties.

- Order
- Linearity
- Homogeneous
- Constant coefficients

Definition 4.2 (Order)

*[22] The **order** of a differential equation is the highest derivative order that appears in the equation* ♣

Definition 4.3 (Linearity)

*If the function F is linear in the dependent variable y and their derivatives $y, y^{'}, y^{''}, ..., y^{(n)}$, the differential equation is said to be **linear.** The general n^{th} order linear differential equation can be written as:*

$$a_0(x)\,y+a_1(x)\,y^{'}+a_2(x)\,y^{''}+.....a_{n-1}(x)\,y^{(n-1)}+a_n(x)\,y^{(n)}=b(x)$$

Where $a_n(x)\neq 0$. Otherwise, the equation is called nonlinear. ♣

- The functions $a_0(x), a_1(x), ..., a_{n-1}(x), a_n(x)$ are called the **variable coefficients**.
- The n^{th} order linear differential equation has **constant coefficients** if the functions $a_0(x)$, $a_1(x), ..., a_{n-1}(x), a_n(x)$ are constants.
- The n^{th} order linear differential equation is **homogeneous**, if and only if $b(x)=0$

$$a_0(x)\,y+a_1(x)\,y^{'}+a_2(x)\,y^{''}+.....a_{n-1}(x)\,y^{(n-1)}+a_n(x)\,y^{(n)}=0$$

Otherwise, the equation is called **non-homogeneous.**

Example 4.2

Classify each of the differential equations listed below by indicating the order of each equation and determining whether it is linear or nonlinear, homogeneous or non-homogeneous, with constant coefficients or with variable coefficients.

$$y^{''} + xy^{'}y = 0 \tag{4.2}$$

$$\frac{d^4u}{dt^4} - 4\frac{d^2u}{dt^2} + 5u = \sin t \tag{4.3}$$

$$\left(\frac{dy}{dx}\right)^3 + 5y = 5x \tag{4.4}$$

$$\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + e^x y = x^3 \tag{4.5}$$

$$y^{''} + \cos xy^{'} + \sqrt{x-1}y = 0 \tag{4.6}$$

The differential equation (4.2) is second order, is nonlinear.

The differential equation (4.3) is 4^{th}order, is nonhomogeneous, is linear, constant coefficients.

The differential equation in (4.4) is first order, is nonlinear.

The differential equation (4.5) is 3^{rd} order, is nonhomogeneous, is linear, variable coefficients.

The differential equation (4.6) is second order, is homogeneous, is linear, have variable coefficients.

4.2 Ordinary Differential Equations

4.2.1 Solution of Ordinary Differential Equations

Definition 4.4

[22] Any real-valued function $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies the following equation over the interval I is called a solution of the differential equation

$$F\left[x, \psi, \psi^{'}, \psi^{''}, ..., \psi^{(n)}\right] = 0$$

♣

Example 4.3

The function f defined for all real x by

$$f(x) = 5\sin x + 4\cos x$$

is a solution of the differential equation

$$y^{''} + y = 0 \tag{4.7}$$

Note that f has a second derivative for all real x

We have

$$\begin{aligned} f^{'}(x) &= 5\cos x - 4\sin x \\ f^{''}(x) &= -5\sin x - 4\cos x \end{aligned}$$

Substituting $f(x) = y$, $f^{''}(x) = y^{''}$, in the equation 4.7 , we obtain

$$-5\sin x - 4\cos x + 5\sin x + 4\cos x = 0$$

Which holds for all real x

4.3 First Order Differential Equations

A first order differential equation is an equation of the form [7, 22]

$$F\left(x, y, y^{'}\right) = 0 \tag{4.8}$$

4.3.1 Linear Equation

The first-order differential equation (4.8) is linear if it can be written in the form

$$a_1(x)\, y^{'} + a_0(x)\, y = b(x)\,, \text{ where } a_1(x) \neq 0$$

4.3.1.1 Linear homogeneous equation

[9, 12] A first order homogeneous linear differential equation is one of the form

$$y^{'} + a(x)\, y = 0 \tag{4.9}$$

Theorem 4.1 (Solution of linear homogeneous equation)

[22, 28] If the function a is continuous on open interval I, then the general solution of the homogeneous equation (4.9) is

$$y = Ke^{-\int a(x)dx} \tag{4.10}$$

♡

Proof we can solve the homogeneous equation (4.9) in the usual way

$$\begin{aligned}
\frac{dy}{dx} &= -a(x)y \\
\frac{1}{y}dy &= -a(x)dx \\
\int \frac{1}{y}dy &= -\int a(x)dx \\
\ln|y| &= -A(x) + C \\
y &= \pm e^{-A(x)+C} \\
y &= Ke^{-A(x)}
\end{aligned}$$

where $A(x)$ is a primitive of $a(x)$ i.e $A(x) = \int a(x)dx$

Example 4.4

Solve the differential equation given by

$$\frac{dy}{dx} + y\sin x = 0$$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= -y\sin x \Rightarrow \frac{-dy}{y} = \sin x dx \\
&\Rightarrow \int \frac{dy}{y} = -\int \sin x dx \\
\ln y &= \cos x + C \\
y &= ke^{\cos x}
\end{aligned}$$

4.3.1.2 Linear non-homogeneous equation

[9, 12] A first order non-homogeneous linear differential equation has the standard form

$$y' + a(x)y = b(x) \tag{4.11}$$

Where $b(x) \neq 0$

Property *If the functions* a, b *are continuous, then the equation*

$$y^{'} + a\left(x\right)y = b\left(x\right)$$

has solutions given by $y_G = y_H + y_P$, *where* y_H *is the solution of a related (homogeneous) equation and* y_P *is a particular solution of the non-homogeneous differential equation*

- **How to find the general solution** y_H

  y_H is the solution of homogeneous equation, which given in theorem 4.1 ie

$$y_H = Ke^{-\int a(x)dx}$$

- **How to find the particular solution** y_P

The **variation of constant** method is a general method that can be used to find the particular solution of a differential equation, by replacing the constant K in the solution y_H of a related (homogeneous) equation by function $K\left(x\right)$, in other word , we put $y_P = K\left(x\right)e^{-\int a(x)dx}$ and determining this function $K\left(x\right)$ by derivative of y_P and substituting in the equation 4.11, finally we can obtain the expression of particular solution as

$$y_P = e^{-\int a(x)dx}\int b\left(x\right)e^{\int a(x)dx}dx$$

Example 4.5

Solve the following differential equation

$$y^{'} - 4xy = -4x^3 \tag{4.12}$$

Solution

The solution of the last equation is given by $y_G = y_H + y_P$

$y_H = ?$

First we solve the homogeneous equation $y^{'} - 4xy = 0$

$$\begin{aligned} y^{'} &= 4xy \Rightarrow \frac{y^{'}}{y} = 4x \\ &\Rightarrow \int \frac{dy}{y} = \int 4x dx \\ &\Rightarrow \ln|y| = 2x^2 + C \\ &\Rightarrow y_H = Ke^{2x^2} \end{aligned}$$

$y_P = ?$

We can find the particular solution, using **Variation of constant** method

$$\begin{aligned} y_P &= K(x) e^{2x^2} \Rightarrow \\ y_P^{'} &= K^{'}(x) e^{2x^2} + K(x) 4xe^{2x^2} \end{aligned} \tag{4.13}$$

Determining $K(x)$ by substituting the expression of y_P into the equation 4.12,

$$\begin{aligned} K^{'}(x) e^{2x^2} + K(x) 4xe^{2x^2} &= 4xK(x) e^{2x^2} - 4x^3 \\ &\Rightarrow K^{'}(x) e^{2x^2} = -4x^3 \\ &\Rightarrow K^{'}(x) = -4x^3 e^{-2x^2} \\ &\Rightarrow K(x) = \int -4x^3 e^{-2x^2} dx \end{aligned}$$

Using integration by parts we get

$$K(x) = \left(x^2 + \frac{1}{2}\right) e^{-2x^2}$$

So,

$$y_P = x^2 + \frac{1}{2}$$

Finally , we obtain the solution of (4.12)

$$y = Ke^{2x^2} + x^2 + \frac{1}{2}$$

4.3.2 Nonlinear Equation

4.3.3 Bernouli equation

A Bernoulli differential equation can be written in the following standard form:

$$y' + a(x)\,y = b(x)\,y^n$$

Where a, b are given continuous functions.

If $n \geq 2$, the equation is first order nonlinear. However, it can be converted into a linear equation by changing the unknown function accordingly.

4.3.3.1 Steps to solve Bernouli equation

Divide the Bernoulli equation by y^n

$$\frac{y'}{y^n} + \frac{a(x)}{y^{n-1}} = b(x) \tag{4.14}$$

Bernoulli equations can be made linear by making the substitution $v = \frac{1}{y^{n-1}}$

Differentiating,

$$v' = (1-n)\,\frac{y'}{y^n}$$

Thus

$$\frac{y'}{y^n} = \frac{v'}{1-n}$$

We substitute this last equation into the equation (4.14) we get

$$\frac{v'}{1-n} + a(x)\,v = b(x)$$

We obtain the linear equation in dependent variable v as

$$v' + (1-n)\,a(x)\,v = (1-n)\,b(x)$$

We solve the linear equation for the changed function.

The final step is to transform the altered function back into its original form.

Example 4.6

Find the solutions the differential equation: $y' = -2xy + 2x^3y^3$

Solution

Let the equation

$$y' + 2xy = 2x^3y^3$$

This is a Bernoulli equation with $n = 3$, $a(x) = 2x$, $b(x) = 2x^3$.

Divide this equation by y^3

$$\frac{y^{'}}{y^3} + \frac{2x}{y^2} = 2x^3 \tag{4.15}$$

We use the substitution $v = y^{-2}$

Differentiating,

$$v' = -2y^{-3}y'$$

Then, we have

$$y' = -\frac{1}{2}y^3v'$$

Substituting for y' *in the differential equation (4.15) we get*

$$-\frac{1}{2}v' + 2xv = 2x^3$$

Which is linear equation in variable v

$$v' - 4xv = -4x^3$$

The solution is

$$v = Ke^{2x^2} + x^2 + \frac{1}{2}$$

Substituting $v = 1/y^2$, *we have*

$$\begin{aligned} y^2 &= \frac{1}{v} \\ y^2 &= \frac{1}{Ke^{2x^2} + x^2 + \frac{1}{2}} \\ y &= \pm\sqrt{\frac{1}{Ke^{2x^2} + x^2 + \frac{1}{2}}} \end{aligned}$$

4.3.4 Separable differential equations

[8, 22] A first order differential equation is separable if it can be written in the form

$$\frac{dy}{dx} = a(x)\,b(y) \tag{4.16}$$

Any separable equation can be solved by means of the following theorem.

Property ***[Separation of variables]***

Let $a(x)$ and $b(y)$ be continuous functions on open intervals I and J, respectively, and assume that $b(y) \neq 0$ on J. Let $A(x)$ be a primitive function of $f(x)$ on I and $B(y)$ be a primitive function of $\frac{1}{b(y)}$ on J.

Then a function y solves the differential equation (4.16) if and only if it satisfies the identity

$$B(y) = A(x) + C$$

for all x in the domain of y, where C is a real constant

Follow these steps to solve a Seperable Differential Equation

- Separate the variables

$$\frac{dy}{b(y)} = a(x)\,dx$$

- Apply the integration operator

$$\int \frac{dy}{b(y)} = \int a(x)\,dx$$

- Since $A(x)$ be a primitive function of $f(x)$ and $B(y)$ be a primitive function of $\frac{1}{b(y)}$, then

$$B(y) = A(x) + C$$

Example 4.7

Find all solutions y of the differential equation

$$y' = \frac{y-1}{x+3}$$

Solution

Dividing both sides of the given differential equation by $y-2$ we get

$$\frac{y'}{y-2} = \frac{1}{x+5}$$

By integration we get

$$\begin{aligned} \int \frac{dy}{y-2} &= \int \frac{dx}{x+5} + k, \\ \Longrightarrow & \ln|y-2| = \ln(x+5) + k \end{aligned}$$

Thus $|y-2| = e^k(x+5)$ from which $y-2 = \pm e^{k(x+5)}$. If we let $K = \pm e^k$, we get

$$y = 2 + K(x+5)$$

Then, the general solution is

$$y = 2 + K(x+5),$$

Example 4.8

Solve the differential equation

$$y' = \frac{2y\sin x}{1+2y^2}$$

Solution

$$\frac{dy}{dx} = \frac{2y\sin x}{1+2y^2}$$

$$\begin{aligned} \Longrightarrow \quad \int \frac{(1+2y^2)\,dy}{2y} &= \int \sin x dx \\ \Longrightarrow \quad \int \left(\frac{1}{2y} + y\right) dy &= \int \sin x dx + C \end{aligned}$$

from which we get the solutions:

$$\frac{1}{2}\ln|y| + \frac{y^2}{2} = -\cos x + C$$

where C is an arbitrary constant.

4.3.5 Euler Homogeneous

A first order nonlinear differential equation is an Euler Homogeneous if it has the form

$$y' = F\left(\frac{y}{x}\right) \tag{4.17}$$

Remark

Let the first order nonlinear differential equation

$$y^{'} = f(x, y)$$

If $f(cx, cy) = f(x, y)$, then the equation $y^{'} = f(x, y)$ is Euler Homogeneous.

4.3.5.1 Steps to solve Euler Homogeneous equation

To solve the equation (4.17), we let $v = \frac{y}{x}$

so that,

$$y = xv \text{ and } y^{'} = v + xv^{'}$$

Introducing these expressions into the differential equation for y we get

$$\begin{aligned} v + xv^{'} &= F(v) \\ v^{'} &= \frac{F(v) - v}{x} \\ \frac{dv}{dx} &= \frac{1}{x}(F(v) - v) \\ \frac{dv}{(F(v) - v)} &= \frac{1}{x}dx \end{aligned}$$

Which is separable differential equation

Example 4.9

Find all solutions y of the differential equation $y' = \frac{x^2+3y^2}{2xy}$.

Solution *The equation is Euler homogeneous, since*

$$f(cx, cy) = \frac{c^2x^2 + 3c^2y^2}{2(ct)(cy)} = \frac{c^2(x^2 + 3y^2)}{c^2(2xy)} = \frac{x^2 + 3y^2}{2xy} = f(x, y)$$

Next we compute the function F

$$y' = \frac{x^2 + 3y^2}{2xy} \Longrightarrow y' = \frac{x^2\left(1 + 3\frac{y^2}{x^2}\right)}{x^2\left(2\frac{y}{x}\right)}$$

$$\Rightarrow y' = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right)$$

Now we introduce the change of functions $v = \frac{y}{x}$*,*

$$y' = \frac{1+3v^2}{2v}$$

Since $y = xv$*, then* $y' = v + xv'$*, which implies*

$$v + xv' = \frac{1+3v^2}{2v} \Rightarrow xv' = \frac{1+3v^2}{2v} - v = \frac{1+v^2}{2v}$$

We obtained the separable equation

$$v' = \frac{1}{x}\left(\frac{1+v^2}{2v}\right)$$

We rewrite and integrate it,

$$\begin{aligned} \frac{2v}{1+v^2}v' &= \frac{1}{x} \Rightarrow \int \frac{2v}{1+v^2}dv = \int \frac{1}{x}dx \\ &\Rightarrow \ln(1+v^2) = \ln(x) + C \end{aligned}$$

Then

$$\begin{aligned} \exp\left(\ln(1+v^2)\right) &= \exp\left(\ln(x) + C\right) \\ 1+v^2 &= c_1x \quad \Rightarrow \quad 1+\left(\frac{y}{x}\right)^2 = c_1x \quad \Rightarrow \quad y(t) = \pm x\sqrt{c_1x-1} \end{aligned}$$

4.4 Second Order Differential Equations

The general second order differential equation is of the form [8]

$$F\left(x, y, y^{'}, y^{''}\right) = 0 \tag{4.18}$$

We proceed to study second-order linear equations with constant coefficients

4.5 Second Order Differential Equations with constant coefficients

The second order differential equation is in a standard form:

$$a_2y^{''} + a_1y^{'} + a_0y = b\left(x\right) \tag{4.19}$$

where a_2, a_1, a_0 are constants, and $a_2 \neq 0$

The homogeneous form of (4.19) is the case when $b(x) = 0$

4.5.1 Linear homogeneous equation

We start to finding general solutions to linear homogeneous equations

$$a_2y^{''} + a_1y^{'} + a_0y = 0 \tag{4.20}$$

Property

If the functions y_1 and y_2 are any two (linearly independent) solutions of the homogeneous linear second order equation

$$a_2y^{''} + a_1y^{'} + a_0y = 0 \tag{4.21}$$

then the linear combination $y_H = c_1y_1(t) + c_2y_2(t)$ is the general solution of the above equation, where c_1, c_2 are constants.

Remark

The functions $y_1(x)$ and $y_2(x)$ are linearly independent if one is not a multiple of the other, in other words $\frac{y_1(x)}{y_2(x)} \neq Cte$

Solving an homogeneous second order ODE

To solve a differential equation in the above form (4.21), we assume a general solution

$$y = e^{rx}$$

of the given differential equation, where r is a constant to be determined, and follow the given steps:

Differentiating we find

$$y = e^{rx} \Rightarrow y^{'} = re^{rx} \Rightarrow y^{''} = r^2e^{rx}$$

Substitution into the differential equation (4.21) yields

$$a_2r^2e^{rx} + a_1re^{rx} + a_0e^{rx} = 0 \Rightarrow e^{rx}\left(r^2 + a_1r + a_0\right) = 0$$

Since e^{rx} can never be zero, so

$$a_2r^2 + a_1r + a_0 = 0 \tag{4.22}$$

This algebraic equation is called the **characteristic equation** of the differential equation.

In solving the characteristic equation (4.22), the following three possibilities, depending on the sign of the discriminant $\Delta = a_1^2 - 4a_2a_0$

1. If $\Delta > 0$, then we have r_1 and r_2 as two real roots to the characteristic equation

$$r_1 = \frac{-a_1 - \sqrt{\Delta}}{2a_2}, \; r_2 = \frac{-a_1 + \sqrt{\Delta}}{2a_2}$$

   In this case the general solution of the linear homogeneous equation (4.21) is

$$y_H = c_1e^{r_1x} + c_2e^{r_2x}$$

2. If $\Delta = 0$, we have one root, $r = \frac{-a_1}{2a_2}$

   In this case the general solution of the equation (4.21) is

$$y_H = c_1e^{r_1x} + c_2xe^{r_1x}$$

3. If $\Delta < 0$, the roots are distinct conjugate complex numbers r_1 and r_2

$$r_1 = \frac{-a_1 - i\sqrt{|\Delta|}}{2a_2} = \alpha - i\beta, \; r_2 = \frac{-a_1 + i\sqrt{|\Delta|}}{2a_2} = \alpha + i\beta$$

   In this case the general solution of the equation (4.21) is

$$y_H = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$$

We show the three cases in the following examples:

Example 4.10

Consider the differential equation

$$4y^{''} + 2y^{'} - 2y = 0$$

The characteristic equation is

$$4r^2 + 2r - 2 = 0$$

$\Delta = 36 > 0$, then we have r_1 and r_2 as two real roots

$$r_1 = \frac{-2 - \sqrt{36}}{8} = -1, \; r_2 = \frac{-2 + \sqrt{36}}{8} = \frac{1}{2}$$

the first solution is

$$y_1(x) = e^{-x}$$

The second solution is

$$y_2(x) = e^{\frac{x}{2}},$$

So, a general solution is

$$y_H(x) = c_1 e^{-x} + c_2 e^{\frac{x}{2}}.$$

Example 4.11

Consider the differential equation

$$y^{''} - 8y + 16 = 0$$

The characteristic equation is

$$r^2 - 8r + 16 = 0 \Rightarrow (r-4)^2 = 0$$

$\Delta = 0$, then we have one root, $r = 4$

Consequently, the first solution is

$$y_1(x) = e^{4x}$$

The second solution is

$$y_2(x) = xe^{4x},$$

So, a general solution is

$$y_H(x) = c_1 e^{4x} + c_2 x e^{4x}.$$

Example 4.12

Consider solving

$$y'' - 6y' + 13y = 0.$$

The characteristic equation is

$$r^2 - 6r + 13 = 0.$$

$\Delta = -16 < 0$

the roots are distinct conjugate complex numbers r_1 and r_2

$$\begin{aligned} r_1 &= \frac{-a_1 - i\sqrt{|\Delta|}}{2a_2} = \frac{6 - i\sqrt{16}}{2} = 3 - 2i \\ r_2 &= \frac{-a_1 + i\sqrt{|\Delta|}}{2a_2} = \frac{6 + i\sqrt{16}}{2} = 3 + 2i \end{aligned}$$

The first solution is

$$y_1(x) = e^{(3-2i)x}$$

The second solution is

$$y_2(x) = e^{(3+2i)x}$$

The general solution of the equation is given by

$$y_H = e^{3x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

4.5.2 Linear non-homogeneous equation

[22] A second order non-homogeneous linear differential equation has the standard form given above as:

$$a_2 y^{''} + a_1 y^{'} + a_0 y = b\left(x\right) \tag{4.23}$$

The general solution of (4.23) is

$$y_G = y_H + y_P$$

Where y_H is the solution of a related (homogeneous) equation (4.21) and y_P is a particular solution of the non-homogeneous differential equation.

The method we discussed in the previous section can be used to determine y_H value since it is the solution of the homogeneous differential equation.

$$y_H = c_1 y_1(t) + c_2 y_2(t)$$

How to find the particular solution y_P

To find the particular solution y_P of equation 4.23 there are two ways

- **First method** (**variation of constants**)

The variation of constants consists of replacing the constants c_1 and c_2 in the solution of a related (homogeneous) equation by functions $c_1(x)$ and $c_2(x)$ and determining what these functions must be to satisfy the original non-homogeneous equation.

Differentiating and substitution into the differential equation (4.23) yields

$$\begin{cases} c_1^{'}(x)\, y_1 + c_2^{'}(x)\, y_2 = 0 \\ c_1^{'}(x)\, y_1^{'} + c_2^{'}(x)\, y_2^{'} = \frac{b(x)}{a_2} \end{cases}$$

Example 4.13

Solve the differential equation

$$y^{''} + y = \sin 2x \tag{4.24}$$

Solution

The general solution of (4.24) is

$$y_G = y_H + y_P$$

Where y_H is the solution of a related (homogeneous) equation and y_P is a particular solution of the non-homogeneous differential equation

First we solve the homogeneous equation

$$y^{''} + y = 0$$

The characteristic equation is

$$r^2 + 1 = 0.$$

$\Delta < 0$

The roots are distinct conjugate complex numbers r_1 and r_2

$$r_1 = -i,\ r_2 = +i$$

The general solution of the homogeneous equation is given by

$$y_H = C_1 \cos x + C_2 \sin x$$

Using the method of variation of constants

$$y_P(x) = C_1(x) \cos x + C_2(x) \sin x$$

The functions $C_1(x)$ and $C_2(x)$ can be determined from the following system of equations:

$$\begin{cases} C_1'(x) \cos x + C_2'(x) \sin x = 0 \\ C_1'(x)(\cos x)' + C_2'(x)(\sin x)' = \sin 2x \end{cases}$$

Then

$$\begin{cases} C_1'(x) \cos x + C_2'(x) \sin x = 0 \\ C_1'(x)(-\sin x) + C_2'(x) \cos x = \sin 2x \end{cases}$$

We can express the derivative $C_1'(x)$ from the first equation:

$$C_1'(x) = -C_2'(x) \frac{\sin x}{\cos x} \tag{4.25}$$

Substituting this in the second equation, we find the derivative $C_2'(x)$:

$$\begin{aligned} \left(-C_2'(x)\frac{\sin x}{\cos x}\right)(-\sin x) + C_2'(x)\cos x &= \sin 2x, \\ &\Rightarrow C_2'(x)\left(\frac{\sin^2 x}{\cos x} + \cos x\right) = \sin 2x, \\ &\Rightarrow C_2'(x)\frac{1}{\cos \text{x}} = \sin 2x \\ &\Rightarrow C_2'(x) = \cos x \sin 2x \end{aligned}$$

Substituting $C_2'(x)$ in the equation (4.25), we find

$$C_1'(x) = -\sin x \sin 2x \tag{4.26}$$

We have $\sin 2x = 2 \sin x \cos x$

$$\begin{cases} C_1(x) = -2 \int \sin^2 x \cdot \cos x dx = -\frac{2}{3} \sin^3 x \\ C_2(x) = -2 \int \cos^2 x \sin x dx = -\frac{2}{3} \cos^3 x \end{cases}$$

Thus, the particular solution to the differential equation can be written as:

$$\begin{aligned} y_P(x) &= C_1(x)\cos x + C_2(x)\sin x \\ &= -\frac{2}{3}\sin^3 x \cdot \cos x - \frac{2}{3}\cos^3 x \cdot \sin x \end{aligned}$$

then, the general solution of the non-homogeneous equation:

$$\begin{aligned} y_G &= y_H + y_P \\ y_G &= C_1 \cos x + C_2 \sin x + -\frac{2}{3}\sin^3 x \cdot \cos x - \frac{2}{3}\cos^3 x \cdot \sin x \end{aligned}$$

- **Second method**

The solution y_P can be determined using the second method by guessing it based on the form of $b(x)$. The table given below shows the possible particular solution y_P corresponding to each $b(x)$.

$b(x)$	y_P
$P_n(x)\, e^{\lambda x}$	**1.** If λ is not a root of characteristic equation, then: $y_P = (q_0 + q_1 x + ... + q_n x^n)\, e^{\lambda x}$ **2.** If λ is a simple root of characteristic equation, then: $y_P = x\,(q_0 + q_1 x + ... + q_n x^n)\, e^{\lambda x}$ **3.** If λ is not a multiple root of characteristic equation, then: $y_P = x^2\,(q_0 + q_1 x + ... + q_n x^n)\, e^{\lambda x}$
$P_n(x)\, e^{\theta x}\cos\sigma x +$ $+ Q_m(x)\, e^{\theta x}\cos\sigma x$	**1.** If $\theta + i\sigma$ is not a root of characteristic equation, then: $y_P = A_n(x)\, e^{\theta x}\cos\sigma x + B_n(x)\, e^{\theta x}\sin\sigma x$ **2.** If $\theta + i\sigma$ is a simple root of characteristic equation then: $y_P = \big(A_n(x)\, e^{\theta x}\cos\sigma x + B_n(x)\, e^{\theta x}\sin\sigma x\big)\cdot x$

Example 4.14

Solve the differential equation

$$y^{''} - 3y^{'} + 2y = e^{2x} \tag{4.27}$$

Solution

The general solution of this equation is

$$y_G = y_H + y_P$$

We solve the homogeneous equation

$$y'' - 3y' + 2y = 0$$

The characteristic equation is

$$r^2 - 3r + 2 = 0.$$

$\Delta = 1 > 0$

Then we have r_1 and r_2 as two real roots

$$r_1 = 2,\ r_2 = 1$$

Hence, the general solution of the homogeneous equation is given by

$$y_H(x) = C_1 e^{2x} + C_2 e^x$$

where C_1, C_2 are constant numbers.

Find a particular solution of the non homogeneous differential equation.

Since e^{2x} is one of the solutions of the homogeneous equation, we look for the particular solution in the form

$$y_P = Axe^{2x}$$

The derivatives are given by

$$\begin{aligned} y'_P &= \left(Axe^{2x}\right)' = Ae^{2x} + 2Axe^{2x} = (A + 2Ax)e^{2x} \\ y''_P &= \left[(A + 2Ax)e^{2x}\right]' = 2Ae^{2x} + (2A + 4Ax)e^{2x} = (4A + 4Ax)e^{2x}. \end{aligned}$$

Substituting the function y_P and its derivatives in the differential equation yields:

$$\begin{aligned} (4A + 4Ax)e^{2x} - 3(A + 2Ax)e^{2x} + 2Axe^{2x} &= e^{2x} \\ \text{Then, } A &= 1 \end{aligned}$$

Thus, the particular solution to the differential equation can be written in the form:

$$y_P = xe^{2x}$$

The general solution of the non homogeneous equation:

$$\begin{aligned} y_G &= y_H + y_P \\ &= C_1e^{2x} + C_2e^x + xe^{2x} \end{aligned}$$

Chapter 4 Exercise

1. **Exercise 1**

   Solve the following first order differential equations with separable variables:

   (a). $2y - xy' = 0$

   (b). $(1 + x^2)\, dy = ydx$

   (c). $y = xy^{'} + y'^{4}$

   (d). $yy' + x = 0$

2. **Exercise 2**

   Solve the following first order differential equations :

   (a). $y' - y = 2e^x$

   (b). $y' + \frac{y}{x} = \ln(x)$

   (c). $y' - 2y = \cos(x) + 2\sin(x)$

3. **Exercise 3**

   Solve the following second-order linear differential equations:

   (a). $y'' - y' - 2y = 0$

   (b). $y'' - y' + y = 0$

   (c). $y'' + 2y' + y = xe^x$

   (d). $y'' + y = 2e^x + \cos(x)$

Chapter 5 Functions of Several Variables

Introduction

- ❏ *Vector valued functions*
- ❏ *Domain and Range*
- ❏ *Graphs*
- ❏ *Limit of a function of two variables*
- ❏ *Continuity for functions of two variables*
- ❏ *Derivatives of a function of two variables*
- ❏ *Total differential*
- ❏ *Double integrals*
- ❏ *Triple Integrals*

5.1 Multivariable vector-valued functions

Definition 5.1

[16] A multivariable vector-valued function is a function $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ has the form

$$\begin{aligned} f:\mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ (x_1, x_2, ..., x_n) &\longmapsto \begin{pmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_m(x_1, x_2, ..., x_n) \end{pmatrix} \end{aligned}$$

Where n and m are positive integers, and $f_1, f_2, ..., f_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ are real-valued functions of several variables. ♣

Remark

We consider several cases of functions with several variables, according to the values of n and m,

- Where $m = 1$, the function f is called a real-valued functions of n variables if assigns to each element of Ω a unique element of $\mathbb{R}$, $f:\Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, where the domain Ω is a subset of $\mathbb{R}^n$. So, for each $(x_1, x_2, ..., x_n)$ in Ω, the value of f is a real number $f(x_1, x_2, ..., x_n)$.

Only real-valued functions of several variables will be considered in this chapter.

5.2 Domain and Range of real-valued functions of several variables

Definition 5.2

*The **domain** of definition of a function f is the set of all possible input $P = (x_1, x_2, ..., x_n)$ of $\mathbb{R}^n$ on which the function f makes sense, it denoted by D_f*

$$D_f = \{P \in \mathbb{R}^n / \ f(x_1, x_2, ..., x_n) \in \mathbb{R}\}$$

♣

Example 5.1

Determine the domain of each of the following

1. $f(x,y) = y\sqrt{1-x^2}$
2. $f(x,y) = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$
3. $g(x,y) = \ln(9 - x^2 - 9y^2)$

Solution

1. $f(x,y) = y\sqrt{1-x^2}$

   The domain D_f are the points (x,y) in the plane defined by:

$$\begin{aligned} D_f &= \left\{(x,y) \in \mathbb{R}^2 / \ 1 - x^2 \geq 0, \ y \in \mathbb{R}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 / \ x^2 \leq 1, \ y \in \mathbb{R}\right\} \end{aligned}$$

   that means $-1 \leq x \leq +1$, $-\infty < y < +\infty$, i.e. the points in the plane between and including the lines $x = 1$, and $x = -1$. It is shown in Figure 5.1.

2. $f(x,y) = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$$\begin{aligned} D_f &= \left\{(x,y) \in \mathbb{R}^2 / \ 1 - x^2 > 0, \ 1 - y^2 \geq 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 / \ x^2 < 1, \ y^2 \leq 1\right\} \end{aligned}$$

   The domain D_f are the points (x,y) where $-1 < x < +1$, and $-1 \leq y \leq +1$, it shown in

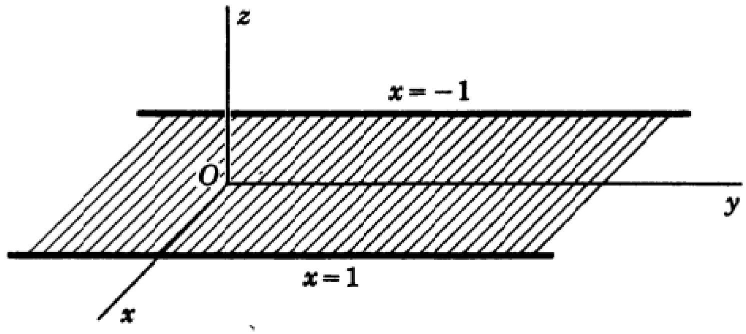


Figure 5.1: Domain of the function $f(x, y) = y\sqrt{1 - x^2}$.

Figure 5.2

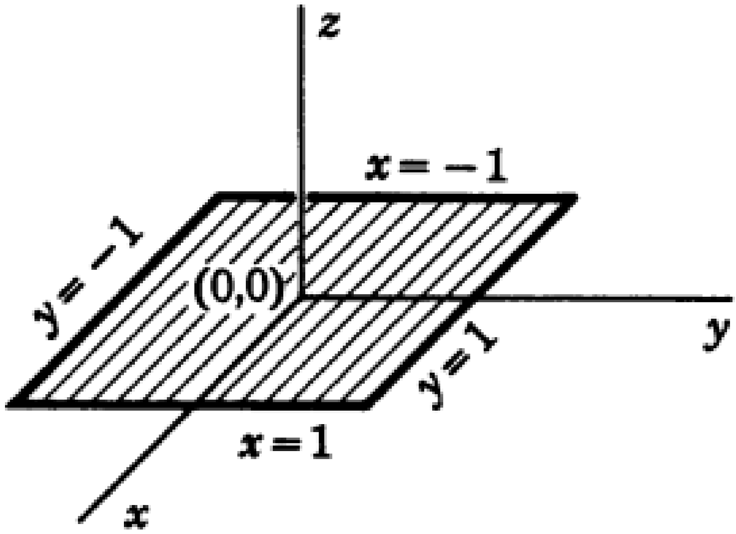


Figure 5.2: Domain of the function $f(x, y) = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

3. $g(x, y) = \ln\left(9 - x^2 - 9y^2\right)$

$$\begin{aligned} D_g &= \left\{(x, y) \in \mathbb{R}^2 / \ 9 - x^2 - 9y^2 > 0\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 / \ \frac{x^2}{9} + y^2 < 1\right\} \end{aligned}$$

Therefore, the domain of $g(x, y)$ is the points interior to an ellipse. See figure 5.3

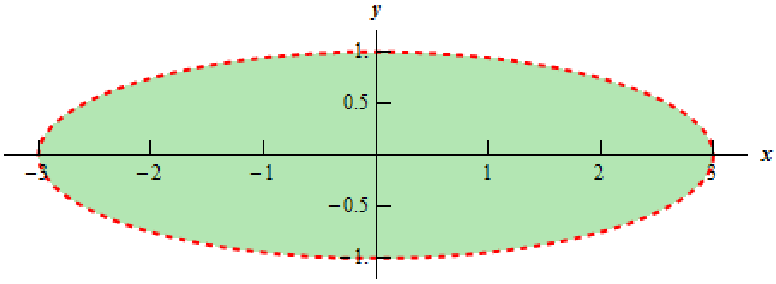


Figure 5.3: Domain of the function $g(x, y) = \ln(9 - x^2 - 9y^2)$

5.3 Graphs of real-valued functions of several variables

Definition 5.3

*Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued functions of n variables. The **graph** of f is the set of point in $\mathbb{R}^{n+1}$ denoted by G_f*

$$G_f = \left\{(x_1, x_2, ..., x_n, x_{n+1}) \in \mathbb{R}^{n+1} / \ x_{n+1} = f(x_1, x_2, ..., x_n)\right\}$$

♣

Example 5.2

$f(x, y) = 1 - \frac{1}{2}(x^2 + y^2)$

The graph of this function is shown in the Figure 5.4

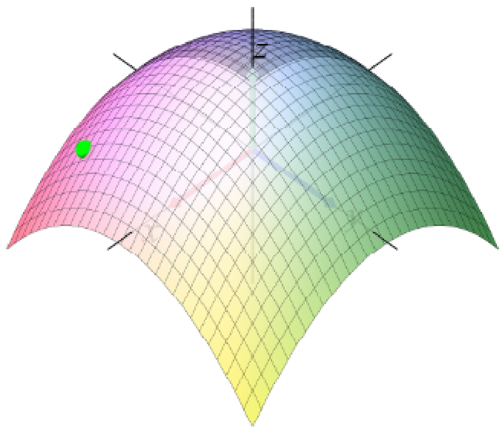


Figure 5.4: Graph of the function $f(x, y) = 1 - \frac{1}{2}(x^2 + y^2)$

Example 5.3

$f(x, y) = \cos(3x) \cdot \sin(3y)$

The graph of this function is shown in the Figure 5.5

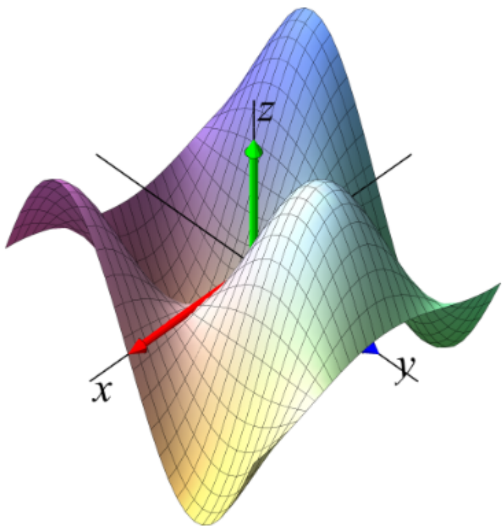


Figure 5.5: Graph of the function $f(x, y) = \cos(3x) \cdot \sin(3y)$

5.4 Limit of a function of two variables

Definition 5.4

[23] Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is l, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = l$$

if for each $\varepsilon > 0$, there exists a small enough $\delta > 0$, such that for all (x, y) in the domain of f

$$\forall\varepsilon > 0,\ \exists\delta > 0,\ \forall(x, y) \in D_f : \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - l| < \varepsilon$$ ♣

Example 5.4

$$\lim_{(x,y)\to(3,4)} \frac{2xy}{x^2+y^2} = \frac{2\cdot3\cdot4}{3^2+4^2} = \frac{24}{25}$$

Property ***[Basic Limit Properties of Functions of Two Variables]***

Let f and g be functions with

$$\lim_{(x,y)\to(a,b)} (f(x, y) = L_1, \lim_{(x,y)\to(a,b)} g(x, y) = L_2$$

The following limits hold.

- $\lim_{(x,y)\to(a,b)}(f(x, y) + g(x, y)) = L_1 + L_2$
- $\lim_{(x,y)\to(a,b)}(f(x, y) - g(x, y)) = L_1 - L_2$

- $\lim_{(x,y)\to(a,b)}(cf(x,y)) = cL_1$
- $\lim_{(x,y)\to(a,b)}(f(x,y)g(x,y)) = L_1L_2$
- $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L_1}{L_2}$ *for* $L_2 \neq 0$
- $\lim_{(x,y)\to(a,b)}(f(x,y))^n = L_1^n$

5.5 Continuity for functions of two variables

The definition of continuity for functions of two variables is similar to that of functions of one variable.

Definition 5.5

[23] Let f be a function of two variables, x and y, let $(a,b) \in D_f$.

- *A function is continuous at a point (a,b) if:*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- *We say f is continuous on D_f if f is continuous at every point (a,b) in D_f.* ♣

Example 5.5

Show that the function

$$f(x,y) = \frac{-x+4y}{x+y+1}$$

is continuous at point $(7,-2)$.

Solution

$$\begin{aligned} D_f &= \left\{(x,y) \in \mathbb{R}^2 / \ 1+x+y \neq 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 / \ x+y \neq -1\right\} \end{aligned}$$

In this example, $a = 7$ and $b = -2$.

$f(a,b) \in D_f$ *because* $7 - 2 \neq -1$. *Furthermore,*

$$f(a,b) = f(7,-2) = \frac{-7+4(-2)}{7+(-2)+1} = \frac{-15}{6}.$$

$\lim_{(x,y)\to(7,-2)} f(x,y)$ *exists.*

Remark

- The sum of continuous functions is continuous
- The product of continuous functions is continuous
- The composition of continuous functions is continuous

5.6 Derivatives of a function of two variables

Definition 5.6

Let $f : D_f \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of two variables, and $(a, b) \in D_f$

- *The partial derivative of f with respect to x , written as $\frac{\partial f}{\partial x}$, or f_x is defined:*

$$f_x^{'}(a,b) = \frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

- *The partial derivative of f with respect to y , written as$\frac{\partial f}{\partial y}$, or f_y is defined:*

$$f_y^{'}(a,b) = \frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

♣

5.6.1 Calculating partial derivatives

The intuitive idea of computing a partial derivative $\frac{\partial f}{\partial x}$ is: Using the usual way of differentiating $f(x, y)$ with respect to x, we calculate using y as a constant, similarly, the partial derivative of f with respect to y is performed while holding x as a constant.

Example 5.6

Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions

1. $f(x, y) = sin(x^2y - 5x + 3)$
2. $g(x, y) = \ln(x^2 + y^2 + 9)$

Solution

1. To calculate $\frac{\partial f}{\partial x}$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y - 5x + 3)\right] \\ &= (2xy - 5)\cos(x^2y - 5x + 3) \end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, treat the variable x as a constant. Then differentiate $f(x, y)$ with respect to y

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y - 5x + 3)\right] \\ &= x^2\cos(x^2y - 5x + 3)\end{aligned}$$

2. To calculate $\frac{\partial g}{\partial x}$, treat the variable y as a constant. Then differentiate $g(x, y)$ with respect to x

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\ln\left(x^2 + y^2 + 9\right)\right] \\ &= \frac{2x}{(x^2 + y^2 + 9)}\end{aligned}$$

To calculate $\frac{\partial g}{\partial y}$, treat the variable x as a constant. Then differentiate $g(x, y)$ with respect to y

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\ln\left(x^2 + y^2 + 9\right)\right] \\ &= \frac{2y}{(x^2 + y^2 + 9)}\end{aligned}$$

5.7 Total differential

Definition 5.7

Let f be a function of two variables x and y

*The **differential** , also called the **total differential** of f , is defined as*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

♣

Definition 5.8

Let f be a function of two variables x and y, $(x_0, y_0) \in D_f$

*If f is differantiable at the point (x_0, y_0) , then the **differential** of f at (x_0, y_0), is defined as:*

$$df\,(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\,dx + \frac{\partial f}{\partial y}(x_0, y_0)\,dy$$

♣

Property

Let f and g be two differantiable functions of two variables The following properties hold.

- $d\,(f + g)\,(x_0, y_0) = df\,(x_0, y_0) + dg\,(x_0, y_0)$
- $d(c \cdot f\,(x_0, y_0)) = c \cdot d(f\,(x_0, y_0))$
- $d\,(f \cdot g)\,(x_0, y_0) = df\,(x_0, y_0) \cdot g\,(x_0, y_0) + f\,(x_0, y_0) \cdot dg\,(x_0, y_0)$

Example 5.7

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of two variables, defined as:

$$f(x,y) = x^2y^2 - 5xy + 3x$$

then

$$\frac{\partial f}{\partial x}(x,y) = 2xy^2 - 5y + 3$$

$$\frac{\partial f}{\partial y}(x,y) = 2yx^2 - 5x$$

The differential of f , is given by:

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \left(2xy^2 - 5y + 3\right) dx + \left(2yx^2 - 5x\right) dy \end{aligned}$$

5.8 Double integrals

Definition 5.9

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of two variables, let D be a closed bounded region in $\mathbb{R}^2$, We denote the double integral of the function f over D by

$$\int\int_D f(x,y)\, dxdy$$

♣

5.8.1 Basic properties of the Integral

Property

Let D be a closed bounded set

$$D = \left\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\right\}$$

Let f , $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be two continuous bounded functions on D. Then if α and β are any constants, we have:

- $\int\int_D (\alpha f + \beta g)(x,y)\, dxdy = \alpha \int\int_D f(x,y) + \beta \int\int_D g(x,y)$.

- $\forall (x,y) \in D,\, f \geq 0 \Rightarrow \int\limits_D \int f(x,y)\, dx \geq 0.$
- *If* $D = D_1 \cup D_2$ *and* $D_1 \cap D_2 = \emptyset$, $\int\limits_D \int f(x,y)\, dxdy = \int\limits_{D_1} \int f(x,y)\, dxdy + \int\limits_{D_2} \int f(x,y)\, dxdy.$
- *If* $f(x,y) \leq g(x,y) \Rightarrow \int\limits_D \int f(x,y) \leq \int\limits_D \int g(x,y)\, dxdy.$
- $\left| \int\limits_D \int f(x,y) \right| \leq \int\limits_{D_1} \int |f(x,y)|\, dxdy$

5.8.2 Integrals over rectangular regions

Property ***[Fubini's theorem 1]*** *Let D be a closed bounded set*

$$D = \left\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\right\}$$

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous bounded function on D. We define an integral for a function f over the rectangular region D as

$$\int\limits_D \int f(x,y)\, dxdy = \int\limits_a^b \left(\int\limits_c^d f(x,y)\, dy \right) dx = \int\limits_c^d \left(\int\limits_a^b f(x,y)\, dx \right) dy$$

The notation $\int\limits_a^b \left(\int\limits_c^d f(x,y)\, dy \right) dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int\limits_c^d \left(\int\limits_a^b f(x,y)\, dx \right) dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant.

Example 5.8

Use Fubini's theorem 1 to evaluate the double integral of f over the rectangular region $D = [0,1] \times [0,4]$,

$$f(x,y) = 3x^2 - y$$

Solution

First integrate with respect to y and then integrate with respect to x :

$$\begin{aligned} \int_0^1 \int_0^4 \left(3x^2 - y\right) dydx &= \int_0^1 \left(\int_0^4 \left(3x^2 - y\right) dy \right) dx \\ &= \int_0^1 \left[3x^2 y - \frac{y^2}{2} \Big|_{y=0}^{y=4} \right] dx \\ &= \int_0^1 \left(12x^2 - \frac{16}{2} \right) dx = 4x^3 - 8x|_{x=0}^{x=1} = 4. \end{aligned}$$

First integrate with respect to x and then integrate with respect to y :

$$\begin{aligned}\int_0^4\int_0^1 \left(3x^2-y\right)dxdy &= \int_0^4\left(\int_0^1\left(3x^2-y\right)dx\right)dy\\ &= \int_0^4\left[x^3-xy|_{x=0}^{x=1}\right]dy\\ &= \int_0^4 (3-y)dy = 3y-\frac{y^2}{2}\Big|_{y=0}^{y=4} = 4\end{aligned}$$

So,

$$\int_0^1\int_0^4 f\left(x,y\right)dydx = \int_0^4\int_0^1 f\left(x,y\right)dxdy$$

Remark

If $f\left(x,y\right)=g\left(x\right)h\left(y\right)$, where g and h are continuous on $[a,b]$ and $[c,d]$ then:

$$\iint_D f\left(x,y\right)dxdy = \int_a^b g\left(x\right)dx\int_c^d h\left(y\right)dy.$$

Example 5.9

Evaluate the integral $\int_0^1\int_1^2 e^{x-y}dxdy$

$$\int_0^1\int_1^2 e^{x-y}dxdy = \int_0^1 e^x dx\int_1^2 e^{-y}dy = (e-1)\left(e^{-1}-e^{-2}\right)$$

5.8.3 Double integrals over non rectangular regions

Property ***[Fubini's theorem 2]***

Let $[a,b]$ *be an closed bounded interval in* $\mathbb{R}$, h_1, h_2, g_1 *and* g_2 *are continuous valued function on* $[a,b]$.

Let D *be one of the following closed bounded sets*

Type I: $D=\left\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ h_1\left(x\right)\le y\le h_2\left(x\right)\right\}$

Or

Type II: $D=\left\{(x,y)\in\mathbb{R}^2 : g_1\left(y\right)\le x\le g_2\left(y\right),\ c\le y\le d\right\}$

If $f : D\longrightarrow\mathbb{R}$ *is countinuous function then:*

Type I: $\displaystyle\iint_D f\left(x,y\right)dxdy = \int_a^b\left(\int_{h_1(x)}^{h_2(x)} f\left(x,y\right)dy\right)dx$

$$\text{Type II: } \int\int_D f(x,y)\,dxdy = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx \right) dy$$

Remark

A regions D of Type I and Type II are shown in Fig 5.6 and Fig 5.7

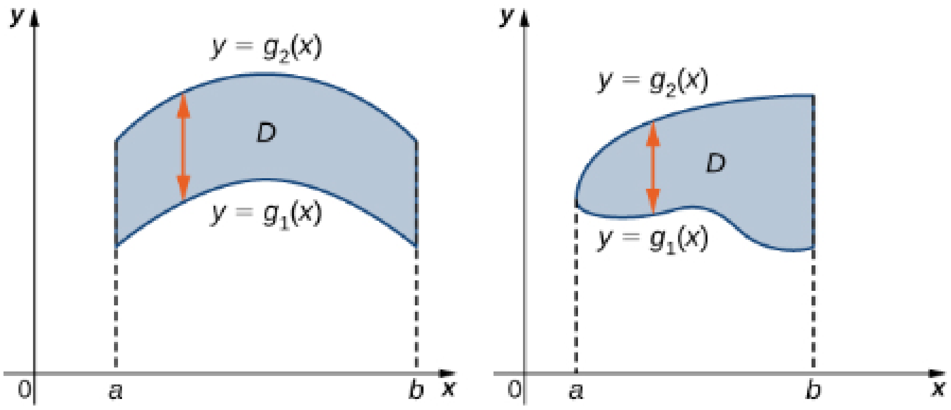


Figure 5.6: A Type I region lies between two vertical lines and the graphs of two functions of x

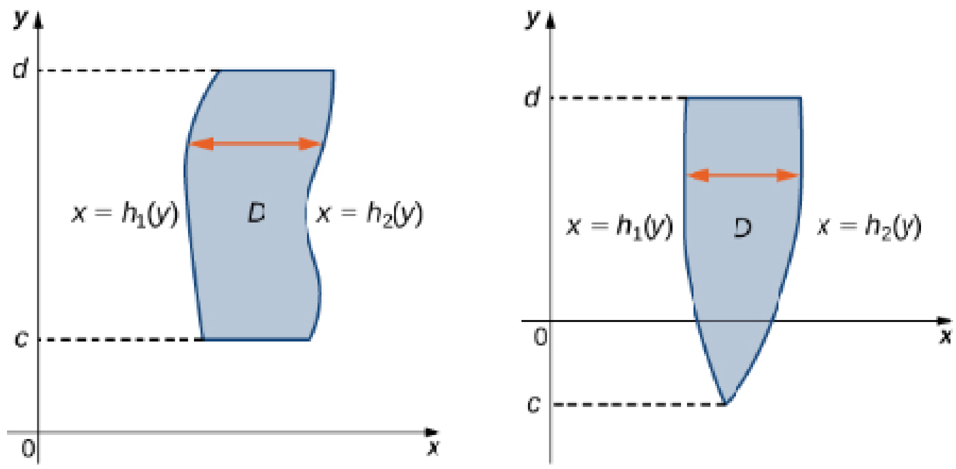


Figure 5.7: A Type II region lies between two horizontal lines and the graphs of two functions of y

Example 5.10

Evaluate the integral

$$\iint_D \cos(xy)dxdy, \quad D = \left\{(x,y) \in \mathbb{R}^2 : 2 \le x \le 4,\ 0 \le xy \le \frac{\pi}{2}\right\}$$

Solution

$$D = \left\{(x,y) \in \mathbb{R}^2 : 2 \le x \le 4,\ 0 \le y \le \frac{\pi}{2x}\right\}$$

$$\int\int_D \cos(xy)dxdy = \int_2^4 \int_0^{\pi/2x} \cos(xy)\,dydx = \int_2^4 \left[\frac{\sin xy}{x}\right]_0^{\pi/2x} dx = \int_2^4 \frac{dx}{x}$$
$$= \ln 4 - \ln 2.$$

5.8.4 Double integral with variable substitution

Definition 5.10

Let D and A be are two closed bounded sets, and φ a bijective differentiable function with continuous partial derivatives:

$$\varphi : A \longrightarrow D$$
$$(u,v) \longmapsto [x(u,v),\ y(u,v)]$$

Let the Jacobian matrix of partial derivatives of φ at the point (u,v)

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

If $\det J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \neq 0$, *then:*

$$\int\int_D f(x,y)dxdy = \int\int_A f[(x(u,v), y(u,v)]\,|\det J|\,dudv$$

♣

Example 5.11

Evaluate the integral

$$\iint_D \cos(xy)dxdy, \quad D = \left\{(x,y) \in \mathbb{R}^2 : 2 \leq x \leq 4,\ 0 \leq xy \leq \frac{\pi}{2}\right\}$$

We use the following variable substitution:

$$\begin{cases} u = xy \\ v = x \end{cases} \Rightarrow \begin{cases} x = v \\ y = \frac{u}{v} \end{cases}$$

$0 \leq u \leq \frac{\pi}{2}, 2 \leq v \leq 4;$

$$J = \begin{pmatrix} 0 & 1 \\ 1/v & -u/v^2 \end{pmatrix}, \quad detJ = \frac{-1}{v} \neq 0$$
$$|\det J| = \frac{1}{v}$$

$$\int\int_D \cos(xy)dxdy = \int_2^4 \int_0^{\pi/2} \frac{1}{v}\cos u du dv = \int_0^{\pi/2} \cos u du \int_2^4 \frac{dv}{v}$$
$$= [\sin u]_0^{\pi/2}[\ln v]_2^4 = \ln 4 - \ln 2.$$

Example 5.12

$\iint_D \sqrt{x^2+y^2}dxdy, \quad D = \{(x,y) \in \mathbb{R}^2_+ : 1 \leq x^2+y^2 \leq 4\}.$

$$D = \{(x,y) \in \mathbb{R}^2_+ : x^2+y^2 \geq 1\} \cap \{(x,y) \in \mathbb{R}^2_+ : x^2+y^2 \leq 4\}$$

We use polar substitution:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad 1 \leq r \leq 2,\ 0 \leq \theta \leq \frac{\pi}{2},$$
$$detJ = r \neq 0$$

$$\iint_D \sqrt{x^2+y^2}dxdy = \int_1^2 \int_0^{\pi/2} r^2 dr d\theta \int_1^2 r^2 dr \int_0^{\pi/2} d\theta = \frac{7\pi}{6}$$

5.9 Triple Integrals

Now that we know how to integrate over a two-dimensional region we need to move on to integrating over a three-dimensional region. We used a double integral to integrate over a two-dimensional region and so it shouldn't be too surprising that we'll use a triple integral to integrate over a three dimensional region. The notation for the general triple integrals is:

$$\iiint_E f(x,y,z)dV$$

Let's start simple by integrating over the box

$$B = [a,b] \times [c,d] \times [r,s]$$

Note that when using this notation we list the x's first, the y's second and the z's third.

The triple integral in this case is

$$\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz$$

Note that we integrated with respect to x first, then y, and finally z here, but in fact there is no reason to the integrals in this order.

Example 5.13

Evaluate the following integral.

$$\iiint_B 8xyz\,dV \quad , \quad B = [2,3] \times [1,2] \times [0,1]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\begin{aligned}
\iiint_B f(x,y,z)dV &= \int_1^2 \int_2^3 \int_0^1 8xyz\,dz\,dx\,dy \\
&= \int_1^2 \int_2^3 \left[4xyz^2\right]_0^1 dx\,dy \\
&= \int_1^2 \int_2^3 4xy\,dx\,dy \\
&= \int_1^2 \left[2x^2y\right]_2^3 dy \\
&= \int_1^2 10y\,dy = 15
\end{aligned}$$

There are six different possible orders to do the integral in and which order you do the integral in will depend upon the function and the order that you feel will be the easiest.

- First case, we define the region E as follows

$$E = \{(x,y,z)/x \in [a,b], u_1(x) \le y \le u_2(x),\ v_1(x,y) \le z \le v_2(x,y)\}$$

In this case we will evaluate the triple integral as follows

$$\iiint_E f(x,y,z)dxdydz = \int_a^b \left(\int_{u_1(x)}^{u_2(x)} \left(\int_{v_1(x,y)}^{v_2(x,y)} f(x,y,z)dz\right) dy\right) dx$$

- Second case we define the region E as follows

$$E = \{(x,y,z)/y \in [a,b], u_1(y) \le x \le u_2(y),\ v_1(x,y) \le z \le v_2(x,y)\}$$

In this case we will evaluate the triple integral as follows

$$\iiint_E f(x,y,z)dxdydz = \int_a^b \left(\int_{u_1(y)}^{u_2(y)} \left(\int_{v_1(x,y)}^{v_2(x,y)} f(x,y,z)dz\right) dx\right) dy$$

- Third case we define the region E as follows

$$E = \{(x,y,z)/z \in [a,b], u_1(z) \le x \le u_2(z),\ v_1(x,z) \le y \le v_2(x,z)\}$$

In this case we will evaluate the triple integral as follows

$$\iiint\limits_E f(x,y,z)dxdydz = \int_a^b \left(\int_{u_1(z)}^{u_2(z)} \left(\int_{v_1(x,z)}^{v_2(x,z)} f(x,y,z)dy \right) dx \right) dz$$

- Fourth case we define the region E as follows

$$E = \{(x,y,z)/z \in [a,b], u_1(z) \leq y \leq u_2(z),\ v_1(y,z) \leq x \leq v_2(y,z)\}$$

In this case we will evaluate the triple integral as follows

$$\iiint\limits_E f(x,y,z)dxdydz = \int_a^b \left(\int_{u_1(z)}^{u_2(z)} \left(\int_{v_1(y,z)}^{v_2(y,z)} f(x,y,z)dx \right) dy \right) dz$$

- Fifth case we define the region E as follows

$$E = \{(x,y,z)/y \in [a,b], u_1(y) \leq z \leq u_2(y),\ v_1(y,z) \leq x \leq v_2(y,z)\}$$

In this case we will evaluate the triple integral as follows

$$\iiint\limits_E f(x,y,z)dxdydz = \int_a^b \left(\int_{u_1(y)}^{u_2(y)} \left(\int_{v_1(y,z)}^{v_2(y,z)} f(x,y,z)dx \right) dy \right) dz$$

- Sixth case we define the region E as follows

$$E = \{(x,y,z)/x \in [a,b], u_1(x) \leq z \leq u_2(x),\ v_1(x,z) \leq y \leq v_2(x,z)\}$$

In this case we will evaluate the triple integral as follows

$$\iiint\limits_E f(x,y,z)dxdydz = \int_a^b \left(\int_{u_1(x)}^{u_2(x)} \left(\int_{v_1(x,z)}^{v_2(x,z)} f(x,y,z)dy \right) dz \right) dx$$

Example 5.14

Evaluate the following integral.

$$\iiint\limits_E \frac{x}{\sqrt{y-x^2}} dxdydz$$

$$E = \left\{(x,y,z) \in \mathbb{R}^3,\ z \in [1,2],\ 0 \leq x \leq \sqrt{y},\ 0 \leq y \leq z^2\right\}$$

Solution

$$\iiint_E \frac{-x}{\sqrt{y-x^2}}dxdydz = \int_1^2 \left(\int_0^{z^2}\left(\int_0^{\sqrt{y}} \frac{-x}{\sqrt{y-x^2}}dx\right)dy\right)dz$$
$$= \int_1^2 \left(\int_0^{z^2} \left[-\sqrt{y-x^2}\right]_0^{\sqrt{y}} dy\right)dz$$
$$= \int_1^2 \left(\int_0^{z^2} \sqrt{y}dy\right)dz = \int_1^2 \left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^{z^2} dz$$
$$= \frac{2}{3}\int_1^2 z^3 dz = \frac{1}{6}\left[z^4\right]_1^2 = \frac{5}{2}$$

Chapter 5 Exercise

1. **Exercise 1** Find the definition set of the following function

$$f(x,y) = \frac{2xy}{x^2+y^2}$$

then study the limit at the point $(0,0)$

2. **Exercise 2**

Calculate the partial derivatives of the function defined by

$$U(x,y) = \sin\left(ax+by+cy^2\right)$$

3. **Exercise 3**

Calculate the total derivative of the function

$$U(x,y) = e^{x^2+y^2}\sin^2\left(x+3y^2\right)$$

4. **Exercise 4**

Evaluate the integral

$$\int\int_D x^2 e^{xy} dxdy$$

$$D = \left\{(x,y)\in\mathbb{R}^2 : 0 \le x \le 2,\ \frac{1}{2}x \le y \le 1\right\}$$

Terminology

Zero matrix	المصفوفة الصفرية
Squar matrix	مصفوفة مربعة
Diagonal matrix	قطر المصفوفة
Identity matrix	المصفوفة الواحدية
Upper Triangular matrix	مصفوفة مثلثية علوية
Lower Triangular matrix	مصفوفة مثلثية سفلية
Symmetric matrix	مصفوفة تناظرية
Special matrices	مصفوفات خاصة
Inverse of a matrix	معكوس مصفوفة
Matrix row	سطر مصفوفة
Matrix column	عمود مصفوفة
Equality of two matrices	تساوي المصفوفات
Transpose of a matrix	منقول مصفوفة
Trace of a square matrix	اثر مصفوفة
Augmented matrix	مصفوفة موسعة
Sub matrix	مصفوفة جزئية
Determinants	المحددات
Cofactor matrix	المصفوفة المرافقة
Minor matrix	المفوفة الصغرى
Rank of a matrix	رتبة مصفوفة
Elementary Transformations	تحويلات اولية
Equivalent matrix	مصفوفات متكافئة
Linear equations	معادلة خطية
System of Linear equations	جمل معادلات خطية
Variables	متغيرات
Unknowns	المجاهيل
Coefficients	المعاملات
Constants	الثوابت
Solution set	مجموعة الحلول
Homogeneous linear system	جمل خطية متجانسة
Unique solution	حل وحيد
Infinitely many solution	مالانهاية من الحلول
Infinitely many solution	مالانهاية من الحلول
Elementary operations	العمليات الاولية
No solution	لاتوجد حلول
Inconsistent system	جمل متعارضة
Matrix form	الشكل المصفوفي
Cramer`s rule	قاعدة كرامر
Matrix inversion	مقلوب مصفوفة
elimination method	طريقة الاختزال
Elementary operations	العمليات الاولية
Equivalent systems	الجمل المتكافئة
Primitive functions	الدوال الاصلية
Derivative functions	مشتق الدوال

Constant of integration	ثابت المكاملة
Continuous function	تابع مستمر
Elementary functions	دوال معروفة
Indefinite integrals	تكامل غير محدود
Definite integrals	تكامل محدود
Linearity of the integral	خطية التكامل
Linearity of the integral	خطية التكامل
Limits of integration	حدود التكامل
integration Methods	طرق التكامل
Integration by Substitution	التكامل بتحويل متغير
Integration by Parts	التكامل بالتجزئة
Partial fractions	الكسور الجزئية
Trigonometric functions	دوال مثلثية
Dfiferential equations	المعادلات التفاضلية
Independent variable	متغير مستقل
Order	درجة
Linearity	الخطية
Homogeneous	متجانسة
Non homogeneous	غير متجانسة
Variable coefficients	المعاملات المتغيرة
Constant coefficients	معاملات ثابتة
General solution	حل عام
Particular solution	حل خاص
Variation of constant	تغيير الثابت
Nonlinear equation	معادلة غير خطية
Bernouli equation	معادلة بارنولي
Separable dfiferential equation	معادلة تفاضلية منفصلة
Euler Homogeneous	اولر المتجانسة
Characteristic equation	المعادلة المميزة
Several variables	متعدد المتغيرات
Domain	مجال التعريف
Range	مجموعة الصور
Graph	بيان
Limit of a function	نهاية الدالة
Multivariable function	متعدد المتغيرات
vector valued functions	دوال شعاعية
Limit of a function	نهاية دالة
Continuity	الاستمرار
Partial derivatives	المشتقات الجزئية
Dfiferential	التفاضل
Total dfiferential	التفاضل التام
Double integrals	التكامل المزدوج
Bounded region	مجال محدود
Jacobian matrix	المصفوفة اليعقوبية

Bibliography

[1] Reza Adhami et al. *Fundamental Concepts in Electrical and Computer Engineering with Practical Design Problems*. Universal-Publishers, 2007.

[2] Martin Anthony and Michele Harvey. *Linear algebra: concepts and methods*. Cambridge University Press, 2012.

[3] Howard Anton and Chris Rorres. *Elementary linear algebra: applications version*. John Wiley & Sons, 2013.

[4] Howard Anton and Chris Rorres. *Elementary linear algebra: applications version*. John Wiley & Sons, 2013.

[5] B Arnaud et al. "Algèbre, Cours de Mathématiques Première Année". In: *Université de Lille* (2020).

[6] Frank Ayres Jr. *Theory and problems of matrices*. McGraw-Hill, 1962.

[7] William E Boyce, Richard C DiPrima, and Douglas B Meade. *Elementary differential equations and boundary value problems*. John Wiley & Sons, 2021.

[8] Matthew Charnley. "An Introduction for Engineers". In: (2021).

[9] Jeffrey R Chasnov. *Differential Equations for Engineers*. 2019.

[10] Paul Moritz Cohn. *Elements of linear algebra*. Routledge, 2017.

[11] Charles G Cullen. *Matrices and linear transformations*. Courier Corporation, 2012.

[12] Paul Dawkins. *Differential equations*. 2007.

[13] Khattar Dinesh. *The Pearson Guide to Mathematics for The IIt-JEE, 3/E*. Pearson Education India, 2010.

[14] Mariana Zamfir Gavriil Paltineanu Ileana Bucur. "Integral Calculus for Engineers". In: *Springer Nature Singapore* (2022).

[15] O'Regan Gerard. *Guide to Discrete Mathematics: An Accessible Introduction to the History, Theory, Logic and Applications*. Springer, 2016.

[16] http://faculty.bard.edu/bloch/multivariablevectorvalued_notes_gray.pdf. "Integral Calculus for Engineers". In: *Bard college* (2023).

[17] RS Johnson. *Integration and differential equations*. Bookboon, 2012.

[18] Frank Ayres Jr. *Theory and problems of matrices*. McGraw-Hill, 1962.

[19] Kenneth Kuttler. *Linear algebra: theory and applications*. The Saylor Foundation, 2012.

[20] Peter Lancaster and Miron Tismenetsky. *The theory of matrices: with applications*. Elsevier, 1985.

[21] Steven J Leon. *Linear algebra with applications*. Pearson, 2014.

[22] Petra Menz et al. "Calculus Early Transcendentals Integral & Multi-Variable Calculus for Social Sciences". In: (2018).

[23] Robert C Rogers. "The calculus of several variables". In: *Online http://www. math. nagoya-u. ac. jp/˜ richard/teaching/s2016/Ref2. pdf* (2011).

[24] Philip E Sarachik. *Principles of linear systems*. Cambridge University Press, 1997.

[25] RD Sarma. *Basic Applied Mathematics for the Physical Sciences: Based on the syllabus of the University of Delhi University, 3/e*. Pearson Education India.

[26] SS Sastry. *Engineering Mathematics Vol*. PHI Learning Pvt. Ltd., 2008.

[27] Thomas S Shores et al. *Applied linear algebra and matrix analysis*. Vol. 2541. Springer, 2007.

[28] Allan Struthers and Merle Potter. *Differential equations: for scientists and engineers*. Springer, 2019.

[29] V Sundarapandian. *Numerical linear algebra*. PHI Learning Pvt. Ltd., 2008.

[30] Daniel Zelinsky. *A first course in Linear Algebra*. Academic Press, 2014.