People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Mentouri University Constantine 1

Faculty of Exact Sciences

Department of Mathematics

Analysis 1

Handout for a First-Year Undergraduate Course
For Engineering Majors in the Faculty of Technology Sciences

Presented by: Zerimeche Hadjer

Academic Year: 2024-2025

The official course syllabus

SEMESTRE	Intitulé de la matière		Coefficient	Code
S1	Analyse 1		5	ANA-1
VHH	**Cours**	**Travaux dirigés**	**Travaux Pratiques**	
67h30	3h00	1h30	-	

Pré requis :

Notions de base des mathématiques des classes Terminales (ensembles, fonctions, équations, …).

Objectifs de l'enseignement

Cette première matière d'Analyse I est notamment consacrée à l'homogénéisation des connaissances des étudiants à l'entrée de l'université. Les premiers éléments nouveaux sont enseignés de manière progressive afin de conduire les étudiants vers les mathématiques plus avancées. Les notions abordées dans cette matière sont fondamentales et parmi les plus utilisées dans le domaine des Sciences et Technologies.

Contenu de la matière:

Chapitre 1 : Propriétés de l'ensemble R

1. Partie majorée, minorée et bornée.
2. Élément maximum, élément minimum.
3. Borne supérieure, borne inférieure.
4. Valeur absolue, partie entière.

Chapitre 2 : Suites numériques réelles

1. Suites convergentes.
2. Théorèmes de comparaison.
3. Théorème de convergence monotone.
4. Suites extraites.
5. Suites adjacentes.
6. Suites particulières (arithmétiques, géométriques, récurrentes)

Chapitre 3 : Les fonctions réelles à une seule variable

1. Limites et continuité des fonctions
2. Dérivée et différentielle d'une fonction
3. Applications aux fonctions élémentaires (puissance, exponentielle, hyperbolique, trigonométrique et logarithmique)

Chapitre 4 : Développement limité

1. Développement limité
2. Formule de Taylor

3. Développement limité des fonctions

Chapitre 5: Intégrales simples

1 Rappels sur l'intégrale de Riemann et sur le calcul de primitives.

Mode d'évaluation :

Interrogation écrite, devoir surveillée, examen final

Références bibliographiques:

1- K. Allab, Eléments d'analyse, Fonction d'une variable réelle, 1^{re}& 2^{e} années d'université, Office des Publications universitaires.

2- J. Rivaud, Algèbre : Classes préparatoires et Université Tome 1, Exercices avec solutions, Vuibert.

3- N. Faddeev, I. Sominski, Recueil d'exercices d'algèbre supérieure, Edition de Moscou

Acknowledgements

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"If you know, let others light their candles in it."
— Margaret Fuller

This course aims to help university students, particularly in their early years, gain a profound understanding of mathematical analysis and learn how to construct proofs. It focuses on fundamental concepts such as the properties of real numbers, convergence theory, continuity, differentiation, and integration.
This course is designed based on the programs for first-year university students in engineering, sciences, and technology. The course covers the following major concepts:

- Chapter 1 introduces the foundational properties of real numbers: Axioms of the real numbers, supremum, infimum, and upper completeness.
- Chapter 2 explores sequences, discussing the basic concepts of convergence and related applications.
- Chapter 3 examines real-valued functions of a single real variable, focusing on limits, continuity, differentiation, L'Hôpital's Rule, Rolle's and the Mean Value theorems.
- Chapter 4 shows the methodology used to approximate functions using polynomials.
- Finally, Chapter 5 transitions to the theory and techniques of integration.

These concepts equip students with the essential tools needed to solve advanced mathematical problems.
Feedback is welcomed and appreciated.

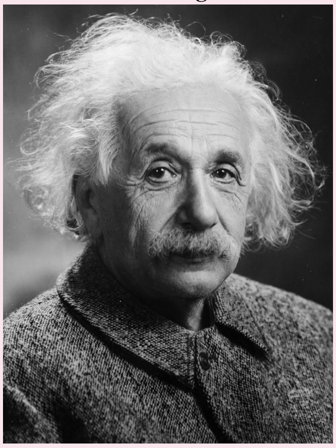
"The art of teaching is the art of assisting discovery"
— Mark Van Doren

Greek alphabet

The Greek alphabet is frequently used in mathematics to represent a wide range of mathematical variables, constants, and symbols. Here are some of the Greek letters commonly used in mathematics and their typical mathematical representations:

Uppercase	Lowercase	Name
A	α	alpha
B	β	beta
Γ	γ	gamma
Δ	δ	delta
E	ϵ	epsilon
Z	ζ	zeta
H	η	eta
Θ	θ	theta
K	κ	kappa
Λ	λ	lambda
M	μ	mu
N	ν	nu
Ξ	ξ	xi
O	o	omicron
Π	π	pi
P	ρ	rho
Σ	σ	sigma
T	τ	tau
Y	υ	upsilon
Φ	ϕ	phi
X	χ	chi
Ψ	ψ	psi
Ω	ω	omega

”The whole of science is nothing more than a refinement of everyday thinking”



Albert Einstein

(1879-1955)

Contents

Preliminaries

Various sorts of numbers

Number sets are collections of numbers with specific properties and characteristics. Below, we introduce some of the most common number sets:

- The empty set (null set) $\varnothing$ denotes the set that contains no elements and is sometimes represented as $\{\}$.
- The set of Natural numbers is represented as $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The set of Integers is represented as $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- The set of Rational numbers is represented as $\mathbb{Q} = \{\frac{p}{q}, \, p \in \mathbb{Z}, \, q \in \mathbb{Z}^*\}$.

  Example 0.1
  - $\frac{1}{4} = 0.25$ (terminating decimals).
  - $\frac{1}{3} = 0.3333$ (repeating decimals).
- Irrational Numbers: These are numbers that cannot be expressed as a fraction, such as $-\sqrt{2}, e, \pi, \sqrt{7}$.
- Real Numbers $\mathbb{R}$: The set of real numbers includes all rational and irrational numbers.
- Complex Numbers $\mathbb{C}$: Complex numbers consist of a real part and an imaginary part. They are written in the form $x + iy$, where x and y are real numbers and i is the imaginary unit ($i^2 = -1$).

Leibniz

(1646-1716)

We have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The universal quantifiers

The notation $\forall$ denotes the universal quantifier.
$\forall x \in \mathbb{R}$ is read: for all real number x.

Peano, Giuseppe

(1858-1932)

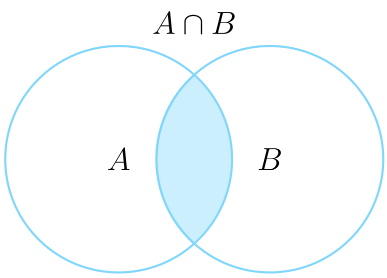
The existential quantifier

The notation $\exists$ denotes the existential quantification.
$\exists x \in \mathbb{R}$ is read: there exists a real number x.
$\exists ! x \in \mathbb{R}$ is read: there exists a unique real number x.

Intersection

The **intersection** of two sets A and B, denoted by $A \cap B$ is the set of elements x that are in both A and B. Mathematically, we represent as follows:

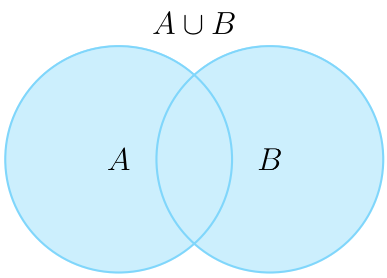
$$A \cap B = \{x \in E : x \in A \text{ and } x \in B\}$$



Union

The **union** of two sets A and B, denoted by $A \cup B$, is the set of elements x that are in A or B (or in both). Mathematically, we express by:

$$A \cup B = \{x \in E : x \in A \text{ or } x \in B\}$$



Inclusion

We say that a set A is **included in** a set B, or that A is a *subset* of B if every element of A is also an element of B. This relationship is denoted as $A \subseteq B$.

$$A \subseteq B \iff \forall x \; (x \in A \implies x \in B) \, .$$

Principle of mathematical induction

Mathematical induction is a method of mathematical proof used to establish that a given statement holds for all natural numbers. The technique consists of two main steps. Let $P(n)$ be a given statement involving the natural number n such that:

- **Base Case:** Show that the statement is true for the first value in the set of natural numbers, usually $n = 1$.
- **Inductive Step:** Assume that the statement is true for some arbitrary natural number k. Then prove that it must also be true for $k + 1$.

If both of these steps are completed, we conclude that $P(n)$ is true for all natural numbers n.

Signum function

Let $x \in \mathbb{R}$. The *sign function*, denoted by $\operatorname{sgn}(x)$, is defined as:

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Chapter 1 Properties of the real numbers

This chapter presents the fundamental properties of the real number system, which form the basis of mathematical analysis. Covers the axioms of real numbers, intervals, absolute values, bounded sets, supremum and infimum, the completeness axiom, the Archimedean principle, and the greatest integer function. These concepts equip students with the essential tools needed for precise reasoning and proofs in mathematics. By the end of this chapter, you will be able to:

- State and apply the axioms of the real numbers.
- Calculate absolute values and determine the boundedness of sets.
- Define, find and use the supremum and infimum of a set, using the completeness axiom.
- Apply and prove properties involving the Archimedean principle and the greatest integer function.

1.1 Axioms for the real numbers

In this section, we introduce the axioms for the real numbers. Recall that in mathematics, axioms are the first principles that are accepted as truths without justification and are used to build mathematical theories. The set of real numbers is denoted by $\mathbb{R}$.

1.1.1 The algebraic properties

There are two operations in $\mathbb{R}$, addition $(+) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and multiplication $(.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

- Commutative law for addition: $x + y = y + x \quad \forall x,\, y \in \mathbb{R}$
- Associative law for addition: $(x + y) + z = x + (y + z) \quad \forall x,\, y,\, z \in \mathbb{R}$
- Existence of identity element (Zero): there exists an element $0 \in \mathbb{R}$ such that

$$0 + x = x + 0 = x, \quad \forall x \in \mathbb{R}$$

- Existence of inverses (Additive Inverse) : $\forall x \in \mathbb{R}\, \exists (-x) \in \mathbb{R}$ such that $x + (-x) = 0$
- Commutative law for multiplication: $x.y = y.x \quad \forall x,\, y \in \mathbb{R}$
- Associative law for multiplication: $(x.y).z = x.(y.z) \quad \forall x,\, y,\, z \in \mathbb{R}$
- Existence of identity element (One): there exists an element $1 \neq 0 \in \mathbb{R}$ such that $1x = x1 = x \quad \forall \in \mathbb{R}$
- Existence of inverses (Multiplicative inverse): for all $x \in \mathbb{R}$ there exists an element $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$
- Distributive law of multiplication over addition: $x(y + z) = x.y + x.z \quad \forall x,\, y,\, z \in \mathbb{R}$

1.1.2 The order properties

$\mathbb{R}$ are totally ordered sets. Now we present the axioms of order:

- $\forall x, y \in \mathbb{R}$ we have $x \leq y$ or $x \geq y$
- $\forall x, y \in \mathbb{R};\ x \leq y$ and $x \geq y$ we have $x = y$
- $\forall x, y, z \in \mathbb{R}$ if $x \leq y$ and $y \leq z$ then $x \leq z$
- $\forall x, y, z \in \mathbb{R}$ if $0 \leq a$ and $x \leq y$ then $ax \leq ay$

As a consequence of these relations, we have the following.

- $\forall x, y \in \mathbb{R}$ if $x \leq y$ then $-x \geq -y$
- $\forall x, y, z \in \mathbb{R}$ if $x \leq y$ and $a \leq 0$ then $ax \geq ay$
- $\forall x \in \mathbb{R}$: $x^2 \geq 0$
- $\forall x \in \mathbb{R}$ if $x > 0$ then $\frac{1}{x} < 1$
- $\forall x, y \in \mathbb{R}$ if $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$

1.2 Interval

If $a, b \in \mathbb{R}$ and $a < b$,

- The open interval:

$$]a; b[= \{x \in \mathbb{R};\ a < x < b\}$$

- The closed interval:

$$[a; b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$$

- The half-open interval:

$$[a; b[= \{x \in \mathbb{R};\ a \leq x < b\}$$

- The half-closed interval:

$$]a; b] = \{x \in \mathbb{R};\ a < x \leq b\}$$

- Infinite intervals are:

$$\begin{gathered}]a; \infty[= \{x \in \mathbb{R};\ x > a\} \\ ]-\infty; b[= \{x \in \mathbb{R};\ x < b\} \\ [a; \infty[= \{x \in \mathbb{R};\ x \geq a\} \\ ]-\infty; b] = \{x \in \mathbb{R};\ x \leq b\} \end{gathered}$$

1.3 The absolute value

We define the absolute value of a real number x, which is denoted by $|x|$ as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Clearly that $|x| = 0$ if and only if $x = 0$ and $0 \leq |x|$ for all $x \in \mathbb{R}$. Some important properties of the absolute value function are presented below:

1. $\forall\, x \in \mathbb{R}$, $\forall\, y \in \mathbb{R}$, we have $|xy| = |x|.|y|$
2. $\forall\, x \in \mathbb{R}$, $|x|^2 = x^2$
3. $\forall\, x \in \mathbb{R}$, $-|x| \leq x \leq |x|$
4. $\forall\, x \in \mathbb{R}$, $\forall\, y \in \mathbb{R}$, we have $|x + y| \leq |x| + |y|$

1.4 Bounded sets

In this section, let us begin with some definitions.

Definition 1.1

Let $A \subset \mathbb{R}$ be a non-empty set is said to be bounded above if there exists a real number M such that $x \leq M$ for all $x \in A$

$$\exists M \in \mathbb{R}, \forall x \in A :\ x \leq M$$

The number M is called an upper bound for the set A.
A set $A \subset \mathbb{R}$ is said to be bounded below if there exists a real number m such that $x \geq m$ for all $x \in A$

$$\exists\, m \in \mathbb{R}, \forall x \in A :\ x \geq m$$

The number m is called a lower bound for the set A.
A set $A \subset \mathbb{R}$ is bounded if it is both bounded above and bounded below.

$$\exists\, m,\ M \in \mathbb{R}, \forall x \in A :\ m \leq x \leq M$$

Example 1.1

For each case, determine if A is bounded above, bounded below, bounded, or unbounded.

- $A =]4, 9[$
- $A =]-\infty, 4]$
- $A = \mathbb{N}^*$

Solution:

- $A =]4, 9[$.
  Every $x \in A$ satisfies $4 < x < 9$, so 4 is a lower bound and 9 is an upper bound. Hence A is bounded.
- $A =]-\infty, 4]$.
  Every $x \in A$ satisfies $x \leq 4$, so 4 is an upper bound. The set has no real lower bound (it is unbounded below), hence A is bounded above.
- $A = \mathbb{N}^* = \{1, 2, 3, \dots\}$.
  Every $n \in \mathbb{N}$ satisfies $n \geq 1$, so 1 is a lower bound, then A is bounded below.

Remark

If A is bounded above with the upper bound a, then any real number greater than a is also an upper bound of A. Similarly, if A is bounded below with a lower bound b, then any real number smaller than b is also a lower bound of A.

Definition 1.2

- *If m is a lower bound for A and $m \in A$ then m is the minimum of A, denoted by* $\min A$.
- *Similarly, if M is an upper bound for A and $M \in A$ then M is the maximum of A, denoted by* $\max A$. *Thus, when they exist,* $\min A$ *and* $\max A$ *belong to A and, for all $x \in A$*

$$\min A \leq x \leq \max A$$

Remark

- Maximum (max): The largest element in the set.
- Minimum (min): The smallest element in the set.

Example 1.2

- Consider$A = [4, 9]$ we have $4 \in A$ then $\min A = 4$ and we have $9 \in A$ then $\max A = 9$
- Consider$A = [4, 9[$ we have $4 \in A$ and $9 \notin A$ then A has a minimum $\min A = 4$ but no maximum.

1.5 Supremum and infimum

The notions of supremum and infimum are fundamental in real analysis, and they play a crucial role in establishing the completeness of the real number system. Let's review the definitions of supremum and infimum, along with some of their fundamental properties. We begin with some definitions.

Definition 1.3

Suppose that $A \subset \mathbb{R}$ is a set of real numbers that are non-empty and bounded.

- *If $M \in \mathbb{R}$ is an upper bound of A such that $M \leq M'$ for every upper bound M' of A, then M is called the supremum of A, denoted as $M = \sup A$. In other words, the supremum of a set is its least upper bound.*
- *If $m \in \mathbb{R}$ is a lower bound of A such that $m \geq m'$ for every lower bound m' of A, then m is called the infimum of A, denoted as $m = \inf A$. In other words, the infimum is its greatest upper bound.*

Remark

- $\sup(A) = \inf\{M;\ M \text{ is an upper bound of } A\}$.
- $\inf(A) = \sup\{m;\ m \text{ is a lower bound of } A\}$.
- If A has a maximum element, then the maximum is equal to the supremum:

$$\max A = \sup A \text{ if } \sup A \in A$$

- If A has a minimum element, then the minimum is equal to the infimum:

$$\min A = \inf A \text{ if } \inf A \in A$$

Example 1.3

1. Let $A = \{\frac{1}{n} : n \in \mathbb{N}^*\}$. Then $\sup A = 1$ belongs to A, so $\max A = 1$. On the other hand, $\inf A = 0$ doesn't belong to A and A has no minimum.
2. Consider the set $A = \{-4, 7, 9\}$. $\sup A = 9$, note that $9 \in A$, then $\max A = 9$. On the other hand, $\inf A = -4$ and $\inf A$ belongs to A, so $\min A = -4$.

Proposition 1.1

If the supremum or infimum of a set A exists, it is unique. If both exist, then $\inf A \leq \sup A$.

Proof

- Suppose that M and M' are supremum of A. Then $M \leq M'$ since M' is an upper bound of A and M is a least upper bound; similarly, $M' \leq M$, so $M = M'$.
- If m and m' are infimum of A, then $m \geq m'$ since m' is a lower bound of A and m is the greatest lower bound; similarly, $m' \geq m$, so $m = m'$.
- The infimum is the greatest lower bound and the supremum is the least upper bound. Therefore, the infimum cannot be greater than the supremum. If $\inf A$ and $supA$ exist, then A is nonempty. Choose $x \in A$. Then $\inf A \leq x \leq \sup A$. It follows that:

$$\inf A \leq \sup A$$

Now, let's present the characterization properties of the supremum and infimum:

Proposition 1.2 (Characterization Property)

Let $A \subset \mathbb{R}$ be a nonempty set that is bounded and $M,\ m \in \mathbb{R}$. We have

$$M = \sup A \iff \begin{cases} \forall x \in A,\ x \leq M \\ \forall \varepsilon > 0,\ \exists x_\varepsilon \in A \text{ such that } x_\varepsilon > M - \varepsilon \end{cases}$$

$$m = \inf A \iff \begin{cases} \forall x \in A,\ x \geq m \\ \forall \varepsilon > 0,\ \exists x_\varepsilon \in A \text{ such that } x_\varepsilon < m + \varepsilon \end{cases}$$

Proposition 1.3

If A, B are nonempty sets, then

$$\sup(A + B) = \sup A + \sup B,\ \inf(A + B) = \inf A + \inf B,$$
$$\sup(A - B) = \sup A - \inf B,\ \inf(A - B) = \inf A - \sup B$$

Proof The set $A + B$ is bounded from above if and only if A and B are bounded from above, so $\sup(A + B)$ exists if and only if both $\sup A$ and $\sup B$ exist. In that case, if $x \in A$ and $y \in B$, then

$$x + y \leq \sup A + \sup B$$

so $\sup A + \sup B$ is an upper bound of $A + B$ and therefore

$$\sup(A+B) \leq \sup A + \sup B$$

To get the inequality in the opposite direction, suppose that $\varepsilon > 0$. Then there exists $x \in A$ and $y \in B$ such that

$$x > \sup A - \frac{\varepsilon}{2} \qquad y > \sup A - \frac{\varepsilon}{2}$$

It follows that

$$x + y > \sup A + \sup B - \varepsilon$$

for every $\varepsilon > 0$, which implies that $\sup(A+B) \geq \sup A + \sup B$. Thus, $\sup(A+B) = \sup A + \sup B$ The proof of the results for $inf(A+B)$ and $inf(A-B)$ are similar.

1.6 Completeness axiom

The completeness axiom is a fundamental concept in the theory of real numbers. It provides a key property that distinguishes the real number system from other number systems, like rational numbers. The completeness of the real numbers ensures the existence of the supremum and infimum. The existence of supremum and infimum is one way to define the completeness of $\mathbb{R}$.

Proposition 1.4 (Completeness axiom)

Every nonempty set of real numbers that is bounded from above has a least upper bound (a supremum). Similarly, every nonempty set of real numbers that is bounded from below has a greatest lower bound (an infimum).

Remark

- The supremum property and the completeness axiom are equivalent.
- The supremum property does not apply to $\mathbb{Q}$.

Example 1.4

Consider the set $A = \{x \in \mathbb{Q}; x^2 < 2\}$

The set A is nonempty and bounded because every element of A satisfies $x^2 < 2$ implying that

$$-\sqrt{2} < x < \sqrt{2}$$

Therefore, $\sqrt{2}$ is an upper bound for A.

Aiming for a contradiction, suppose that $\sup A \in \mathbb{Q}$ exists and $\sqrt{2} \notin \mathbb{Q}$ then

$$\sup A \neq \sqrt{2} \iff \sup A > \sqrt{2} \text{ or } \sup A < \sqrt{2}$$

- If $\sup A < \sqrt{2}$

  So, using the theorem on the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_1 \in \mathbb{Q}$ such that $\sup A < r_1 < \sqrt{2}$

  $$r_1 \in A \iff r_1 \in \mathbb{Q} : r_1^2$$

  Therefore from,

$$\forall x \in A \Rightarrow x \leq \sup A$$

we have $r_1 \in A$ and $\sup A < r_1$
This contradicts the characterization property of supremum.

- If $\sup A > \sqrt{2}$
  So, using the theorem on the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r_2 \in \mathbb{Q}$ such that $\sqrt{2} < r_2 < \sup A$
  Therefore, by the definition of supremum is the smallest upper bound. r_2 is an upper bound, and $r_2 < \sup A$
  which is a contradiction with $\sup A$ is the smallest upper bound. Then $\sup A \notin \mathbb{Q}$.

Since there is no rational least upper bound for A.

1.7 The Archimedean principle

The completeness axiom implies an important property of the real numbers, known as the Archimedean principle. It states that if x and y are real numbers with $x > 0$, then there exists a natural number $n \in \mathbb{N}$ such that

$$nx > y.$$

1.7.1 Application

Consider the set $A = \{4 - \frac{1}{n};\, n \in \mathbb{N}^*\}$
Find the sup and inf if there exist.

Taking $a_n = 4 - \frac{1}{n}$, we have the set A is nonempty, moreover

$$\forall n \in \mathbb{N}^*;\ 3 \leq a_n < 4$$

Since A is bounded. By the completeness axiom, $\sup A$ and $\inf A$ exist.

- $\inf A$: Infimum is the greatest lower bound.

  $$\forall n \in \mathbb{N}^*;\, 4 - \tfrac{1}{n} \geq 3$$

  The set of lower bounds is $]-\infty, 3]$; then $\inf A = 3$.
- $\sup A$: It seems that 1 is the upper bound of A. Using the characterization property of the supremum to prove that 1 is the least upper bound for A.

  $$\sup A = 4 \iff \forall \varepsilon > 0\, \exists n_\varepsilon \in \mathbb{N};\, 4 - \varepsilon < a_{n\varepsilon}$$

  Suppose that: $4 - \varepsilon < a_{n\varepsilon}$ then $4 - \varepsilon < 4 - \frac{1}{n_\varepsilon}$, thus $n_\varepsilon > \frac{1}{\varepsilon}$
  By Archimedean principle, there exists n_ε satisfying the above inequality. So $\sup A = 4$.

1.8 The greatest integer function

Definition 1.4

For real numbers x, the greatest integer function denoted as $[x]$ or $E(x)$ gives the greatest integer not greater than x.

Example 1.5

$E(9,4) = 9,\ E(\pi) = 3,\ E(-9,4) = -10.$

The greatest integer function has several interesting properties:

- If x is a real number, then

$$E(x) \leq x < E(x) + 1$$

- Let x be a real number and let n be an integer. Then

$$E(x+n) = E(x) + n$$

- If $x, y \in \mathbb{R}$ and $x \geq y$, then $E(x) \geq E(y)$.
- $E(x) = x$, if x is an integer.

Chapter 1 Exercises

Exercise 1

Prove the following properties:

- $\forall x, y \in \mathbb{R},\ |x+y| \leq |x| + |y|$.
- $\forall x, y \in \mathbb{R},\ \big||x| - |y|\big| \leq |x-y|$.
- $\forall x, y \in \mathbb{R},\ |x| + |y| \leq |x+y| + |x-y|$.
- $\forall x \in \mathbb{R},\ |x| = \max\{x, -x\}$.

Exercise 2

If the set A is bounded, find $\sup A$, $\max A$, $\inf A$, and $\min A$ if they exist.

$$A = \{x \in \mathbb{R} : 0 < x < 9\}, \quad A = \left\{9 - \frac{1}{n}, n \in \mathbb{N}^*\right\},$$

$$A = \{x \in \mathbb{R} : x^3 > 64\}, \quad A = \left\{\frac{1}{x} : 4 \leq x \leq 9\right\},$$

$$A = \left\{\frac{n+2}{n-1}, n \in \mathbb{N}, n \geq 2\right\}, \quad A = \left\{9 + \frac{1}{n}, n \in \mathbb{N}^*\right\}$$

Exercise 3

Find the sup, max, inf and min of the following sets and prove your answer.

- $A = \left\{\frac{8}{n^2+4};\ n \in \mathbb{N}\right\}$
- $A = \left\{\frac{2n+1}{n+1};\ n \in \mathbb{N}\right\}$

Exercise 4

Suppose that A and B are nonempty and bounded sets of real numbers. Prove that:

- If $A \subset B$, then $\sup A \leq \sup B$ and $\inf B \leq \inf A$
- $\inf(A \cup B) = \min(\inf A, \inf B)$
- $\sup(A \cup B) = \max(\sup A, \sup B)$

Exercise 5

Suppose that A and B are nonempty and bounded sets of real numbers. Prove that:
If $A \cap B \neq \varnothing$, then $A \cap B$ is bounded:

$$\max(\inf A, \inf B) \leq \inf(A \cap B) \leq \sup(A \cap B) \leq \min(\sup A, \sup B)$$

Exercise 6

Prove the following properties:

- $\forall x, y \in \mathbb{R} : x \leq y \Rightarrow E(x) \leq E(y)$
- $\forall x \in \mathbb{R}, n \in \mathbb{N}^* : E(\frac{E(nx)}{n}) = E(x)$

Chapter 2 Sequences

The concept of sequences has its roots in the earliest stages of mathematics, where ordered patterns of numbers were studied long before the development of formal analysis. Ancient Greek mathematicians, such as Pythagoras and Euclid, examined numerical patterns and progressions, including arithmetic and geometric sequences, as part of their investigations into number theory and geometry. Later, mathematicians in the Islamic world expanded on these ideas, exploring series and summations in algebraic contexts.

The modern notion of sequences as a foundation for limits and convergence began to take shape in the 17th century with the work of mathematicians such as Newton and Leibniz, who used infinite sequences and series in the development of calculus. In the 19th century, definitions of limits and convergence, introduced by mathematicians like Cauchy and Weierstrass, provided the formal framework that underpins real analysis today. The main objectives of this chapter are to:

- Present the formal definition of a sequence of real numbers.
- Study bounded sequences and their properties.
- Define convergence of sequences and examine its behavior.
- Explore monotone sequences and the monotone convergence theorem.
- Establish the relationship between limits and inequalities.
- Analyze adjacent sequences, subsequences, and their convergence properties.
- Examine special classes of sequences, including geometric and recursively defined sequences.

2.1 Sequence of real numbers

Suppose that for each positive integer n, we are given a real number u_n. Then the list of numbers

$$u_1, u_2, \ldots, u_n, \ldots$$

is called a sequence. This ordered list is usually written as

$$(u_1, u_2, \ldots) \quad \text{or} \quad (u_n) \quad \text{or} \quad \{u_n\}.$$

Formally, a sequence is defined as follows:

Definition 2.1 (Sequence of real)

A sequence of real numbers is a real-valued function whose domain is the set of natural numbers. We denote a sequence with standard functional notation such as $f : \mathbb{N} \to \mathbb{R}$. it is customary to use subscripts, replace $f(n)$ with u_n, and denote a sequence $\{u_n\}$ or u_1, $u_2 \ldots$.
A natural number n is called an index for the sequence, and the number corresponding to the index n is called the nth term of the sequence.

2.2 Bounded sequence

Definition 2.2

- *A sequence* $\{u_n\}$ *of real numbers is bounded above if*

$$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}\text{: } u_n \leq M$$

- *A sequence* $\{u_n\}$ *of real numbers is bounded below if*

$$\exists m \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}\text{: } u_n \geq m$$

- *A sequence* $\{u_n\}$ *of real numbers is bounded if*

$$\exists M,\, m \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}:\ m \leq u_n \leq M$$

or equivalently

$$\exists B \geq 0 \text{ such that } \forall n \in \mathbb{N}\text{: } |u_n| \leq B$$

Example 2.1

- $u_n = n^2$ is a bounded below, since $\forall n \in \mathbb{N}$, $u_n \geq 0$.
- $u_n = \frac{1}{n}$ is a bounded, since $\left|\frac{1}{n}\right| \leq 1$ for all $n \in \mathbb{N}^*$.

2.3 Sequence convergence

Definition 2.3

A sequence $\{u_n\}$ *converges to the number* $l \in \mathbb{R}$ *if*

$$\forall \varepsilon > 0,\, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \Rightarrow |u_n - l| < \varepsilon$$

We call l *the limit of the sequence. We write:*

$$\lim_{n \to \infty} u_n = l \text{ or } u_n \to l$$

Example 2.2

1. Let the sequence $u_n = \frac{n+1}{4n+1}$.
   We claim $lim_{n\to\infty} u_n = \frac{1}{4}$. To see this, we want to demonstrate that

   $$\forall \varepsilon > 0,\, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \Rightarrow |u_n - l| < \varepsilon$$

   That is

   $$\forall \varepsilon > 0,\, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \Rightarrow \left|\frac{n+1}{4n+1} - \frac{1}{4}\right| < \varepsilon$$

   We must therefore prove the existence of $n_0 \in \mathbb{N}$, which verifies

   $$n \geq n_0 \Rightarrow \left|\frac{n+1}{4n+1} - \frac{1}{4}\right| < \varepsilon$$

We begin by examining the size of the difference, and simplifying it:

$$\left|\frac{n+1}{4n+1} - \frac{1}{4}\right| < \varepsilon \Rightarrow \frac{3}{16n+4} < \varepsilon$$
$$\Rightarrow \frac{3-4\varepsilon}{16\varepsilon} < n$$

The Archimedean property guarantees the existence of n_0. Taking $n_0 = E(\frac{3-4\varepsilon}{16\varepsilon}) + 1$, then we obtain

$$lim_{n\to\infty} u_n = \frac{1}{4}$$

2. Consider the sequence $u_n = \frac{n+1}{n+2}$. We claim $lim_{n\to\infty} u_n = 1$. To see this, we need to prove that

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \Rightarrow |u_n - l| < \varepsilon$$

That is

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \Rightarrow \left|\tfrac{n+1}{n+2} - 1\right| < \varepsilon$$

We must therefore prove the existence of $n_0 \in \mathbb{N}$, which verifies

$$n \geq n_0 \Rightarrow \left|\tfrac{n+1}{n+2} - 1\right| < \varepsilon$$

We begin by examining the size of the difference, and simplifying it:

$$\left|\frac{n+1}{n+2} - 1\right| < \varepsilon \Rightarrow \frac{1}{n+2} < \varepsilon$$
$$\Rightarrow \frac{1}{\varepsilon} - 2 < n$$

The Archimedean property guarantees the existence of n_0. Taking $n_0 = E(\frac{1}{\varepsilon} - 2) + 1$, then

$$\lim_{n\to\infty} u_n = 1$$

Remark

A sequence that converges is said to be **convergent**, and otherwise is said to be **divergent**.

Theorem 2.1

A convergent sequence has a unique limit.

Proof Suppose u_n converges to l_1 and to l_2. So, $lim_{n\to\infty} u_n = l_1$ and $lim_{n\to\infty} u_n = l_2$ where $l_1 \neq l_2$. Firstly, given $u_n \to l_1$, let $\varepsilon = \frac{l_1 - l_2}{4}$

$$\exists n_1 \in \mathbb{N} \text{ such that } \forall n \geq n_1, |u_n - l_1| < \varepsilon.$$

Then, given $u_n \to l_2$,

$$\forall\varepsilon > 0, \exists n_2 \in \mathbb{N} \text{ such that } \forall n \geq n_2, |u_n - l_2| < \varepsilon$$

Consider $n_0 = \max\{n_1, n_2\}$, Then, we have both

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 : |u_n - l_1| < \varepsilon \text{ and } |u_n - l_2| < \varepsilon$$

Now, apply the triangle inequality to the terms $u_n - l_1$ and $u_n - l_2$. Hence

$$\begin{aligned} |l_1 - u_n + u_n - l_2| &\leq |u_n - l_1| + |u_n - l_2| \\ &< 2\varepsilon \\ &= \frac{l_1 - l_2}{2} \end{aligned}$$

This is a contradiction. The assumption that $l_1 \neq l_2$ cannot be true. Therefore, the limit of a convergent sequence is unique, and the proof is complete.

Remark

To show that the sequence u_n is divergent, it is sufficient to demonstrate that it tends to two different values.

Example 2.3

Consider the sequence $u_n = (-1)^n$. This sequence is divergent because

$$u_n = \begin{cases} 1 & \text{if} \quad n = 2k \\ -1 & \text{if} \quad n = 2k+1 \end{cases}$$

divergence is evident because there is no unique limit as n approaches infinity.

Theorem 2.2

If $\{u_n\}$ is convergent, then $\{u_n\}$ is bounded.

Proof Suppose that u_n converges to l. Given $\varepsilon = 1$. Thus, there exists an $n_0 \in \mathbb{N}$ such that $|u_n - l| < 1$ for all $n \geq n_0$. Let

$$B = \max\{|u_1|, |u_2|, \ldots, |u_{n_0-1}|, |l| + 1\}.$$

for all $n \geq n_0$, we have

$$|u_n| = |u_n - l + l| \leq |u_n - l| + |l| < 1 + |l|$$

then for all $n \in \mathbb{N}$ we have

$$|u_n| \leq B.$$

Hence, we have shown that a convergent sequence $\{u_n\}$ is bounded.

Remark

A bounded sequence is not necessarily convergent.

Example 2.4

The sequence $u_n = (-1)^n$ is bounded since $\forall n \in \mathbb{N} : -1 \leq u_n \leq 1$. However, despite being bounded, the sequence is divergent.

2.4 Monotone sequences

Definition 2.4

- *A sequence $\{u_n\}$ is said to be monotone increasing if $\forall n \in \mathbb{N}$, $u_n \leq u_{n+1}$.*
- *A sequence $\{u_n\}$ is said to be monotone decreasing if $\forall n \in \mathbb{N}$, $u_n \geq u_{n+1}$.*
- *If $\{u_n\}$ is either monotone increasing or monotone decreasing, we say $\{u_n\}$ is monotone or monotonic.*

Example 2.5

Let's consider the sequence $u_n = \frac{1}{n}$ for $n \geq 1$. This sequence is monotone decreasing.

2.5 The monotone convergence criterion

Theorem 2.3

Let $\{u_n\}$ be a monotone increasing sequence. Then, $\{u_n\}$ is convergent if and only if $\{u_n\}$ is bounded. Moreover,

$$\lim_{n\to\infty} u_n = l = \sup\{u_n;\, n \in \mathbb{N}\}$$

Proof Firstly, we know that if $\{u_n\}$ is convergent then by the preceding theorem, it is bounded. Now assume that $\{u_n\}$ is bounded. By the completeness axiom, the set $A = \{u_n;\, n \in \mathbb{N}\}$ has a supremum, define $l = \sup\{u_n;\, n \in \mathbb{N}\}$. We claim that $u_n \to l$. We want to prove that.

$$\lim_{n\to\infty} u_n = l.$$

Let $\varepsilon > 0$. Since l is an upper bound for A, $u_n \leq l$ for all n. Since $l - \varepsilon$ is not an upper bound for A, there is an index n_0 for which $u_{n_0} > l - \varepsilon$. Since the sequence is increasing, $u_n > l - \varepsilon$ for all $n \geq n_0$. If $n \geq n_0$ we have

$$l - \varepsilon < u_n \leq l < l + \varepsilon$$

Thus, if $n \geq n_0$ then $|u_n - l| < \varepsilon$. Therefore, $u_n \to l$.

Theorem 2.4

Let $\{u_n\}$ be a monotone decreasing sequence. Then, $\{u_n\}$ is convergent if and only if $\{u_n\}$ is bounded. Moreover,

$$\lim_{n\to\infty} u_n = \inf\{u_n;\, n \in \mathbb{N}\}$$

This proof is similar to the previous theorem.

2.6 Limits and inequalities

In this section, we explore fundamental results concerning the limits of sequences. We begin by examining the interaction between sequences and inequalities. Subsequently, we delve into the relationship between limits and inequalities, introducing the squeeze theorem.

Theorem 2.5 (Squeeze theorem)

Let $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ be sequences such that $\forall n \in \mathbb{N}$,

$$u_n \leq v_n \leq w_n.$$

Suppose that $\{u_n\}$ and $\{w_n\}$ converge and

$$\lim_{n\to\infty} u_n = l = \lim_{n\to\infty} w_n.$$

Therefore, $\{v_n\}$ converges and

$$\lim_{n\to\infty} u_n = l$$

Proof Let $\varepsilon > 0$. The sequences u_n and w_n are convergent and have the same limit l then

$\lim_{n\to\infty} u_n = l$, there exists an $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$|u_n - l| < \varepsilon$$

Since $\lim_{n\to\infty} w_n = l$, $\exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$,

$$|w_n - l| < \varepsilon$$

In particular, we have $l - \varepsilon < u_n$. Similarly, we have that $w_n < \varepsilon + l$.
Putting everything together, we find

$$l - \varepsilon < u_n \leq v_n \leq w_n < l + \varepsilon \implies |v_n - l| < \varepsilon.$$

Choose $n_0 = \max\{n_1, n_2\}$. Then, if $n \geq n_0$, then

$$|v_n - l| < \varepsilon.$$

Therefore, $\{v_n\}$ is convergent and

$$\lim_{n\to\infty} v_n = l$$

Example 2.6

Consider the sequence v_n defined as follows:

$$v_n = \sum_{k=1}^{n} \frac{n^2}{n^3 + k},\ n \geq 1$$

It is clear that for all $n \geq k \geq 1$ we have

$$\frac{n^2}{n^3 + n} \leq \frac{n^2}{n^3 + k} \leq \frac{n^2}{n^3 + 1}$$

then

$$\sum_{k=1}^{n} \frac{n^2}{n^3 + n} \leq \sum_{k=1}^{n} \frac{n^2}{n^3 + k} \leq \sum_{k=1}^{n} \frac{n^2}{n^3 + 1}$$

so

$$n\frac{n^2}{n^3 + n} \leq v_n \leq n\frac{n^2}{n^3 + 1}$$

then

$$\frac{n^3}{n^3 + n} \leq v_n \leq \frac{n^3}{n^3 + 1}$$

Now as

$$\lim_{n\to\infty} \frac{n^3}{n^3 + n} = \lim_{n\to\infty} \frac{n^3}{n^3 + 1} = 1$$

both sequences approach 1. By the squeeze theorem, we get

$$\lim_{n\to\infty} v_n = 1$$

Theorem 2.6 (Linearity and monotonicity of convergence)

Let $\{u_n\}$ and $\{v_n\}$ be convergent sequences of real numbers. Then

1. *For each pair of real numbers α and β, the sequence $\alpha u_n + \beta v_n$ is convergent and*

$$\lim_{n\to\infty} \left[\alpha u_n + \beta v_n\right] = \alpha \lim_{n\to\infty} u_n + \beta \lim_{n\to\infty} v_n$$

2. *If $u_n \leq v_n$ for all n, then*

$$\lim_{n\to\infty} u_n \leq \lim_{n\to\infty} v_n$$

Proof

1. Define

$$\lim_{n\to\infty} u_n = l_1 \text{ and } \lim_{n\to\infty} v_n = l_2$$

Observe that

$$\big|[\alpha u_n + \beta v_n] - [\alpha l_1 + \beta l_2]\big| \leq |\alpha||u_n - l_1| + |\beta||v_n - l_2|, \forall n \in \mathbb{N} \quad (*)$$

Given $\varepsilon > 0$. Choose a natural number n_0 such that

$$|u_n - l_1| < \frac{\varepsilon}{(2+2|\alpha|)} \text{ and } |v_n - l_2| < \frac{\varepsilon}{(2+2|\beta|)}, \forall n \geq n_0$$

From (*) we obtain that

$$\big|[\alpha u_n + \beta v_n] - [\alpha l_1 + \beta l_2]\big| < \varepsilon, \forall n \geq n_0$$

2. Let $u_n \to l_1$ and $v_n \to l_2$, suppose $w_n = v_n - u_n$ and $l = l_2 - l_1$. Since $u_n \leq v_n$ we have $v_n - u_n \geq 0$ then $w_n \geq 0$, by linearity of convergence $w_n \to l$. We must show $l \geq 0$. Given $\varepsilon > 0$, find an $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ we have

$$|w_n - l| < \varepsilon \implies -\varepsilon + l < w_n < \varepsilon + l$$

Also $w_n \geq 0$. In particular, for $n \geq n_0$ we have $0 \leq w_n < l + \varepsilon$ because $\varepsilon > 0$ this would imply that $l \geq 0$.

Therefore,

$$l_1 \leq l_2$$

2.7 Algebraic operations of sequence convergence

Theorem 2.7

Suppose $\lim_{n\to\infty} u_n = l_1$ *and* $\lim_{n\to\infty} v_n = l_2$*. Then,*

1. $\{u_n \cdot v_n\}$ *is convergent and* $\lim_{n\to\infty} u_n v_n = l_1 l_2$.
2. *If* $\forall n \in \mathbb{N}$, $v_n \neq 0$ *and* $l_2 \neq 0$, *then* $\{u_n/v_n\}_n$ *is convergent and*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{l_1}{l_2}.$$

Proof

1. Since $v_n \to l_2$ and $\{u_n\}$ then, it is bounded. In other words, $\exists B \geq 0$ such that $\forall n \in \mathbb{N}$, $|v_n| \leq B$. Then,

$$\begin{aligned} |u_n v_n - l_1 l_2| &= |(u_n - l_1)v_n + (v_n - l_2)l_1| \\ &\leq |u_n - l_1||v_n| + |l_1||v_n - l_2| \\ &\leq B|u_n - l_1| + |l_1||v_n - l_2|. \end{aligned}$$

Therefore,

$$0 \le |u_n v_n - l_1 l_2| \le B|u_n - l_1| + |l_1||v_n - l_2|$$

Since

$$B|u_n - l_1| + |l_1||v_n - l_2| \to 0$$

By the squeeze theorem $\lim_{n\to\infty} |u_n v_n - l_1 l_2| = 0$.

2. We prove that $\frac{1}{v_n} \to \frac{1}{l_2}$. We first prove $\exists m > 0$ such that $\forall n \in \mathbb{N}$, $|v_n| \ge m$. Since $v_n \to l_2$ and $l_2 \ne 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$,

$$|v_n - l_2| < \frac{|l_2|}{2}.$$

By the triangle inequality, $\forall n \ge n_0$,

$$|l_2| \le |v_n - l_2| + |v_n| \le \frac{|l_2|}{2} + |v_n| \implies |v_n| \ge \frac{|l_2|}{2}.$$

Let $m = \min\left\{|u_1|, \ldots, |u_{n_0-1}|, \frac{|l_2|}{2}\right\}$. Then, $\forall n \in \mathbb{N}$, $|v_n| \ge m$.
Therefore,

$$0 \le \left|\frac{1}{v_n} - \frac{1}{l_2}\right| = \frac{|v_n - l_2|}{|v_n||l_2|} \le \frac{1}{m|l_2|}|v_n - l_2|.$$

According the squeeze theorem,

$$\lim_{n\to\infty} \left|\frac{1}{v_n} - \frac{1}{l_2}\right| = 0$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{v_n} = \frac{1}{l_2}$$

Furthermore, by the proof before, it follows that

$$\lim_{n\to\infty} \left(u_n \cdot \frac{1}{v_n}\right) = \frac{l_1}{l_2}$$

Remark

$$\lim_{n\to\infty} (u_n)^k = l^k.$$

Now, we present the definition of sequences converging to infinity, call ∞ the limit of u_n, and write $\lim_{n\to\infty} u_n = \infty$

Definition 2.5 (Limits at infinity)

We say the sequence $\{u_n\}$ converges to infinity if and only if:

$$\lim_{n\to\infty} u_n = +\infty \iff \forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \ge n_0 \implies u_n > A$$
$$\lim_{n\to\infty} u_n = -\infty \iff \forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \ge n_0 \implies u_n < -A$$

2.8 Adjacent sequences

Definition 2.6

Let $\{u_n\}$ and $\{v_n\}$ be two sequences of real numbers. We say that $\{u_n\}$ and $\{v_n\}$ are adjacent if

- $u_n \leq u_{n+1}$ *and* $v_{n+1} \leq v_n$, *for all* $n \in \mathbb{N}$
- *and*

$$\lim_{n\to\infty} (v_n - u_n) = 0$$

Example 2.7

Let $u_n = 9 - \frac{4}{n^2}$ and $v_n = 9 + \frac{4}{n^2}$. Then

- $u_{n+1} - u_n = \frac{-4n^2+4(n+1)^2}{n^2(n+1)} > 0$, u_n is increasing sequence.
- $v_{n+1} - v_n = \frac{-4(n+1)^2+4n^2}{n^2(n+1)} < 0$, v_n is decreasing sequence.
- we have $v_n - u_n = \frac{8}{n^2}$. Thus

$$\lim_{n\to\infty} (v_n - u_n) = \lim_{n\to\infty} \frac{8}{n^2} = 0$$

Then $\{u_n\}$ and $\{v_n\}$ are adjacent.

Proposition 2.1

Two adjacent sequences u_n and v_n converge to the same limit l.

2.9 Subsequence

A subsequence of a sequence $\{u_n\}$ is a sequence formed by taking certain terms from the original sequence, in the same order as they appeared in the original sequence. More precisely, we have the following definition.

Definition 2.7

Informally, a subsequence is a sequence with entries coming from another given sequence. In other words, for a sequence $\{u_n\}$ and a strictly increasing sequence of natural numbers $\{n_k\}$ we call the sequence $\{u_n\}$ whose kth term is u_{nk} a subsequence of $\{u_n\}$.

Example 2.8

Consider the sequence $u_n = (-1)^n$. Then we have the subsequences $u_2 = 1$ and $u_{2n+1} = -1$.

Lemma 2.1

Let $n_1 < n_2 < \cdots < n_k < \ldots$ be an increasing sequence of natural numbers; that is,$n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $k \leq n_{nk}$.

Theorem 2.8

If $\{u_n\}$ converges to l, then any subsequence of u_n will converge to l.

Proof Suppose $u_n \to l$. Let u_{nk} be any subsequence of $\{u_n\}$. We shall prove that $\lim_{n\to\infty} u_{nk} = l$. To do this, Let $\varepsilon > 0$. Since, $\lim_{n\to\infty} u_n = l$, then $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$,

$$|u_n - l| < \varepsilon.$$

Now suppose k is any natural number such that $k \geq n$, then $n_k \geq k \geq n \geq n_0$ by Lemma 2.1. Hence, for all $\varepsilon > 0$ there exists an n_0 such that for all $n_k > n_0$, and implies that

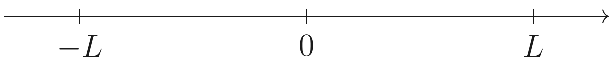
$$|u_{nk} - l| < \varepsilon.$$

Theorem 2.9 (Bolzano-Weierstrass)

Every bounded sequence has a convergent subsequence.

Proof Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $|a_n| \leq L$ for all $n \in \mathbb{N}$.

Step 1: So $-L \leq a_n \leq L$. Note that $[-L, L] = [-L, 0] \cup [0, L]$. Divide the interval $[-L, L]$ into two halves. At least one half must contain infinitely many a_n. Pick one such half and call it I_1. Note that $|I_1| = \frac{1}{2}|[-L, L]| = L$. In fact, say $I_1 = [a_0, a_0 + L]$. So $a_n \in I_1$. There are infinitely many a_n's in I_1. Select one, say a_{n_1}.



Divide the interval I_1 into two halves. At least one half must contain infinitely many a_n. Pick one such half and call it I_2. Note that $|I_2| = \frac{1}{2}|I_1| = \frac{L}{2}$. In fact, say $I_2 = [a_1, a_1 + \frac{L}{2}]$. So $a_n \in I_2$. There are infinitely many a_n's in I_2. Select one, say a_{n_2} with $n_2 > n_1$.

In this way we generate

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots \quad \text{with } n_1 < n_2 < n_3 < \cdots$$

Note that $|I_k| = \frac{L}{2^{k-1}}$. Also, $I_{k+1} \subset I_k$ and $a_{n_k} \in I_k$, $a_{n_{k+1}} \in I_{k+1}$ for all $n \in \mathbb{N}$. Also $a_{n_k} \in S \subseteq [-L, L]$.

Step 2: The sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is monotone increasing and bounded above by L. So it is convergent. Call the limit a.

Step 3: Prove that $a_{n_k} \to a$.

Let $\varepsilon > 0$. Since (a_{n_k}) converges to a, there exists N_1 such that

$$|a_{n_k} - a| < \tfrac{\varepsilon}{2} \quad \text{whenever } n_k \geq N_1.$$

Since $|I_k| = \frac{L}{2^{k-1}} \to 0$, there exists N_2 such that

$$|I_k| < \tfrac{\varepsilon}{2} \quad \text{whenever } k \geq N_2.$$

Thus, the distance from a_{n_k} to a is at most the length of I_k, which converges to zero as $k \to \infty$. Choose $N = \max(N_1, N_2)$. Then $n_k \geq N \implies |a_{n_k} - a| \leq |a_{n_k} - \xi| + |\xi - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, $(a_{n_k}) \to a$.

Remark

If a sequence is not bounded, it must have subsequences diverging to positive or negative infinities.

2.10 Geometric sequence

Definition 2.8

Given real numbers a and r. the real numbers

$$a, ar, ar^2 \dots$$

are said to form a geometric sequence, also known as a geometric progression. a is called the initial term and r is called the common ratio.

The n-th term u_n of a geometric sequence can be expressed using the formula:

$$u_n = ar^{n-1}$$

n is the term number.

Example 2.9

- The numbers $-9, 36, -144, 576, \dots$ form a geometric sequence with initial term -9 and common ratio -4.
- The sequence $u_n = (-1)^n$ is a geometric sequence.

Remark

- To prove that a sequence is geometric, it is sufficient to demonstrate that the ratio $\frac{u_{n+1}}{u_n}$ does not depend on n.

Proposition 2.2

The sequence r^n is convergent if $-1 < r < 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} +\infty & \text{if } r > 1, \\ 1 & \text{if } r = 1, \\ 0 & \text{if } -1 < r < 1. \end{cases}$$

Remark

- We now present some very useful fundamental limits

$$\lim_{n\to\infty} n^r = \begin{cases} +\infty & \text{if } r > 0, \\ 1 & \text{if } r = 0, \\ 0 & \text{if } r < 0. \end{cases}$$

- And

$$\lim_{n\to\infty} (1 + \frac{a}{n})^n = e^a$$

Proposition 2.3

If a and r are real numbers and $r \neq 1$. Then

$$\sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

2.11 Recursively defined sequences

We now give other useful ways to define sequences.

Definition 2.9

Let I be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a real function defined on I. Let $a_0 \in \mathbb{R}$. We call a sequence defined recursively any sequence $\{u_n\}$ defined as

$$\begin{cases} u_0 &= a_0 \\ u_{n+1} &= f(u_n), \quad \forall n \in \mathbb{N} \end{cases}$$

Example 2.10

Consider the recursively defined sequence

$$\begin{cases} u_0 = 1, \\ u_{n+1} = 3 + u_n, \quad \forall n \in \mathbb{N}. \end{cases}$$

Proposition 2.4

If a sequence $\{u_n\}$ defined by the recurrence relation $u_{n+1} = f(u_n)$ converges to a limit l in the domain of D_f, and if the function f is continuous, then we have $f(l) = l$. The value l is called a fixed point of the function

Proposition 2.5

Let $\{u_n\}$ be a sequence defined by the recurrence relation $u_0 = a_0$ and $u_{n+1} = f(u_n)$. If f is increasing, then $\{u_n\}$ is monotonic. More precisely:

- *If $u_1 \geq u_0$, the sequence $\{u_n\}$ is increasing. It converges if and only if it is bounded above; otherwise, it tends to $+\infty$.*
- *If $u_1 \leq u_0$,, the sequence $\{u_n\}$ is decreasing. It converges if and only if it is bounded below; otherwise, it tends to $-\infty$.*

Example 2.11

Define recursively a sequence u_n by:

$$\begin{cases} u_0 &= 1, \\ u_{n+1} &= \sqrt{u_n + 6}, \quad \forall n \in \mathbb{N}. \end{cases}$$

The generating function of the sequence u_n is $f(x) = \sqrt{x+6}$ It is defined over the interval $[-6, +\infty[$. Note that all terms of the sequence $\{u_n\}$ are positive; therefore, this sequence is well-defined and we have $\forall n \in \mathbb{N} : \ u_n \in D_f$. Additionally, we have

$$f'(x) = \frac{1}{2\sqrt{6+x}} > 0$$

indicating that f is increasing. Thus, $\{u_n\}$ is monotonic.

Since $u_1 - u_0 = 1 - \sqrt{7} > 0$ is an increasing sequence.

Let's now demonstrate that $\{u_n\}$ is bounded above. To achieve this, we will show by induction that

$$\forall n \in \mathbb{N}: \; u_n \leq 3$$

- Base case :
  For $n = 0$ we have $u_0 = 1 < 3$
- Inductive step:
  Assume that $u_n \leq 3$ is true and let's show that $u_{n+1} \leq 3$
  We have

$$\begin{aligned} u_n \leq 3 &\Rightarrow u_n + 6 \leq 9 \\ &\Rightarrow \sqrt{u_n + 6} \leq \sqrt{9} \\ &\Rightarrow u_{n+1} \leq 3 \end{aligned}$$

by mathematical induction, we conclude that

$$\forall n \in \mathbb{N}: \; u_n \leq 3.$$

Then we can write: $\forall n \in \mathbb{N} : u_n \in [0,3]$, $\{u_n\}$ is increasing and bounded, therefore, it converges to a limit $l \in [0,3]$.
Since the function f is continuous on $[0,3]$, this limit satisfies $f(l) = l$.
Solving $\sqrt{6+l} = l$, we find $l = -2 \notin [0,3]$ or $l = 3 \in [0,3]$. Then,

$$\lim_{n\to\infty} u_n = 3$$

Chapter 2 Exercises

Exercise 1

Consider the sequences:

$$(1)\, u_n = (1+\frac{1}{n})^n, \qquad (2)\, u_n = \sqrt{n^2+4n} - n,$$

$$(3)\, u_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}, \qquad (4)\, u_n = \frac{n\sin(n)}{n^2+1}$$

- Determine the limit of the sequence u_n as n approaches infinity.
- Using the definition of limit, verify that.

$$\lim_{n\to+\infty} u_n = \frac{4n-1}{2n+1} = 2, \qquad \lim_{n\to+\infty} u_n = \sqrt{n^2+1} - \sqrt{n} = +\infty$$

Exercise 2

Consider the sequence:

$$u_n = \tfrac{1}{n+1} + \tfrac{1}{n+2} + \tfrac{1}{n+3} \cdots + \tfrac{1}{2n}$$

- Prove that the sequence u_n is monotone increasing.
- Prove that the sequence u_n is convergent, and its limit satisfies:

$$\tfrac{1}{2} \leq l \leq 1$$

Exercise 3

Consider the sequence u_n defined by $u_n = \sqrt{n} - E(\sqrt{n})$

- Study the convergence of the subsequence u_{n^2}, u_{n^2+2n}.
- What can you conclude about the nature of the sequence u_n?

Exercise 4

Define recursively a sequence u_n by:

$$\begin{cases} u_0 &= \dfrac{3}{2} \\ u_{n+1} &= (u_n - 1)^2 + 1 \end{cases}$$

- Prove that $\forall n \in \mathbb{N}; \quad 1 < u_n < 2.$
- Prove that u_n is monotone sequence.
- If u_n converges, compute its limit.

Exercise 5

Define recursively a sequence u_n by:

$$\begin{cases} u_0 &= 1, \\ u_{n+1} &= \dfrac{u_n + 1}{2u_n + 3}, \quad \forall n \in \mathbb{N}. \end{cases}$$

- Prove that $\forall n \in \mathbb{N}, u_n > 0.$
- Prove that $\forall n \in \mathbb{N}^*, (u_{n+1} - u_n)(u_{n+1} - u_{n-1}) \geq 0.$
- Conclude that this sequence is monotone.
- Is this sequence convergent? If it is convergent, find its limit.

Exercise 6

Prove that each of the following pair of sequence $(u_n)_{n\in\mathbf{N}}$ and $(v_n)_{n\in\mathbf{N}}$ are adjacent

$$(1)\, u_n = \sum_{n=1}^{n} \frac{1}{k^2+1},\ v_n = u_n + \frac{1}{n} + \frac{1}{2n^2}$$

$$(2)\, u_n = \sum_{n=1}^{n} \frac{1}{k!},\ v_n = u_n + \frac{1}{nn!}$$

Chapter 3 Real functions of one real variable

The functions of a real variable form the basis for many concepts in mathematical analysis and its applications. They are essential tools for describing curves, modeling physical phenomena, and performing mechanical calculations. The main objective of this chapter is to introduce the fundamental properties of real functions and to develop the skills necessary to analyze their behavior.

In particular, we aim to:

- Define functions of a real variable and represent them graphically.
- Study boundedness, monotonicity, symmetry, and periodicity of functions.
- Perform algebraic operations with functions.
- Understand and apply the notions of limits and continuity.
- Explore elementary, trigonometric, inverse trigonometric, and hyperbolic functions.
- Introduce the concept of the derivative and its main properties.
- Find the nth derivative (with $n \geq 1$) of the function, whenever it exists.

3.1 Definition of function and its graph

Definition 3.1

Let $D \subseteq \mathbb{R}$ be a nonempty set. A function f of a real variable is a rule which assigns to each $x \in D$ exactly one $y \in \mathbb{R}$:

$$f : D \longrightarrow \mathbb{R}, \quad x \longmapsto y = f(x).$$

- *The set D is called the **domain** of the function f and is denoted by D_f.*
- *$f(x)$ is called the **image** of x, or the **value** of the function at x.*
- *The set $\{ y = f(x) \mid x \in \mathrm{D}_f \}$ is called the **image** or **range** of f and is denoted by $\mathrm{Im}(f)$ or $f(\mathrm{D}_f)$.*
- *A function is often called a **mapping**.*

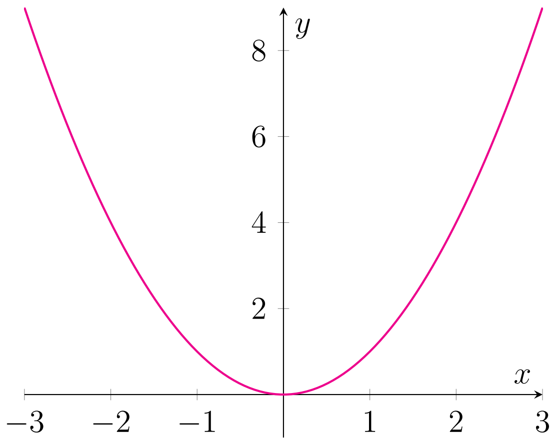
Definition 3.2

*The **graph** of a function f is the set of ordered pairs of real numbers $(x, f(x))$, where $x \in \mathrm{D}_f$. We write*

$$G = \{ (x, f(x)) \in \mathbb{R}^2 \mid x \in \mathrm{D}_f \}.$$

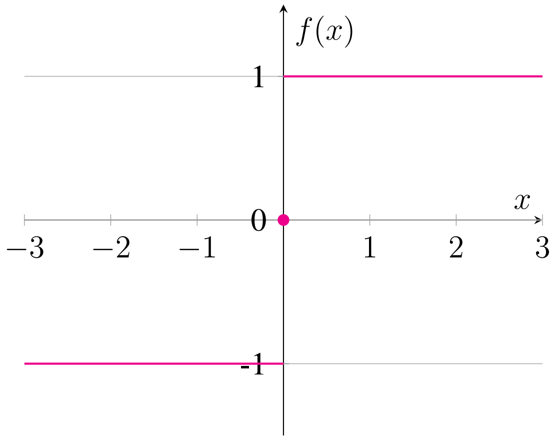
Example 3.1

- The graph of the function $f(x) = x^2$ is



- The graph of the function

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$



3.2 Bounded functions

Definition 3.3

Let $f : D \to \mathbb{R}$

- *A function f is said to be upper bounded if*

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in D\text{: } f(x) \leq M$$

- *A function f is said to be lower bounded if*

$$\exists m \in \mathbb{R} \text{ such that } \forall x \in D\text{: } f(x) \geq m$$

- *A function f is said to be bounded if it is both upper and lower bounded or equivalently*

$$\exists B \geq 0 \text{ such that } \forall x \in D\text{: } |f(x)| \leq B$$

Example 3.2
Show that $f(x) = \frac{x^2-9}{x^2+9}$ is bounded
We see that for any $x \in \mathbb{R}$ we have,

$$\frac{x^2-9}{x^2+9} = 1 - \frac{18}{x^2+9}$$

Holds

$$x^2 \geq 0 \implies x^2+9 \geq 9 \implies \forall x \in \mathbb{R};\ 0 \leq \frac{1}{x^2+9} \leq \frac{1}{9}$$

Then

$$-2 \leq \frac{-18}{x^2+9} \leq 0$$

So,

$$-1 \leq f(x) \leq 1$$

Remark
As we saw in Chapter 2, it is possible to write

$$M = \sup f \iff \begin{cases} \forall x \in D,\ f(x) \leq M \\ \forall \varepsilon > 0,\ \exists x_0 \in D \text{ such that } f(x_0) > M - \varepsilon \end{cases}$$

3.3 Monotonic functions

Definition 3.4
Consider $f : D \to \mathbb{R}$. A function f is said to be:

- *increasing if for any $x_1, x_2 \in D$ such that $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds.*
- *decreasing if for any $x_1, x_2 \in D$ such that $x_1 < x_2$ the inequality $f(x_1) > f(x_2)$ holds.*
- *Functions that are increasing or decreasing are called strictly monotonic.*

Example 3.3
Identify the monotonicity of the following functions:

$$f(x) = x^2; \qquad f(x) = \frac{1}{x}$$

3.4 Even and odd functions

Definition 3.5
Let $f : D \to \mathbb{R}$ is called

- *even if $f(-x) = f(x)$ for any $x \in D$.*
- *odd if $f(-x) = -f(x)$ for any $x \in D$.*

Example 3.4
Decide if the following functions are even or odd.

$$f(x) = \frac{x}{x^2+4} \text{ and } f(x) = \frac{9-x^2}{9+x^2}$$

3.5 Periodic functions

Definition 3.6

Function f is said to be periodic with period $p, p \in \mathbb{R}$, $p > 0$, if,

- *for any $x \in D$ also $x + p \in D$, and*
- *$f(x + p) = f(x)$ for any $x \in D$.*

The best-known periodic functions are trigonometric functions. For example, $sine$ and $cosine$ have the primitive period 2π.

3.6 Operations with functions

The definition below is natural. We add, subtract, multiply, and divide function values of two functions at points where both these functions are defined. Moreover, in the case of division, the divisor must be nonzero.

Definition 3.7

Consider two real function $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Their sum $f + g$, difference $f - g$, product fg and quotient f/g are defined as follows:

- $\forall x \in D : (f + g)(x) = f(x) + g(x)$.
- $\forall x \in D : (f - g)(x) = f(x) - g(x)$.
- $\forall x \in D : (fg)(x) = f(x)g(x)$.
- $\forall x \in D : (\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ *and* $g(x) \neq 0$.
- *Multiplication by a constant function $\lambda \in \mathbb{R}$, $(\lambda f)(x) = \lambda f(x)$.*

Definition 3.8

Consider two real function $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$. The composite function, denoted as $f \circ g$ is defined if and only if $f(D_f) \subset D_g$ and we have

$$x \in D_f \overset{f}{\mapsto} f(x) \in f(D_f) \overset{g}{\mapsto} g(f(x)) = g \circ f(x)$$

$$\forall x \in D_f : \ g \circ f(x) = g(f(x))$$

Remark

In mathematics the symbol $g \circ f$ is read ”g composed with f”.

3.7 Limits and continuity of functions

The concept of limit is one of the most important in mathematical analysis. In this section, we will describe several types of limits of functions of one variable. Using limit we will then introduce continuity, another fundamental concept.

3.7.1 Limits of functions

Definition 3.9

A set $U \subset \mathbb{R}$ is a neighborhood of a point $x \in \mathbb{R}$ if

$$]x-\varepsilon; x+\varepsilon[\subset U$$

Let $\varepsilon > 0$. The open interval $]x-\varepsilon; x+\varepsilon[$ is called a ε-neighborhood of x and denoted $V_\varepsilon(x)$

Definition 3.10

Function $f : D \to \mathbb{R}$ has the limit $l \in \mathbb{R}$ at the point x_0 if and only if:

$$\forall \varepsilon > 0,\ \exists \delta > 0, \text{ such that } \forall x \in D:\ 0 < |x - x_0| < \delta \iff |f(x) - l| < \varepsilon$$

We then say $f(x)$ converges to l as x goes to x_0. We write

$$\lim_{x \to x_0} f(x) = l$$

or

$$f(x) \to l \quad as \quad x \to x_0$$

Example 3.5

For the function f given by $f(x) = 3x + 3$, we have $\lim_{x\to 1} f(x) = 6$. So, $\forall x \in \mathbb{R}$:

$$|f(x) - 6| = |3x + 3 - 6| = 3|x - 1| \iff |x - 1| < \frac{\varepsilon}{3}$$

It is enough to take $\delta = \frac{\varepsilon}{3}$ to have:

$$\forall \varepsilon > 0,\ \exists \delta > 0, \text{ such that } \forall x \in D:\ 0 < |x - 1| < \delta \iff |f(x) - 6| < \varepsilon$$

Theorem 3.1

The limit of a function, if it exists, is unique.

Proof Suppose that f has two distinct limits l_1 and l_2 as x goes to x_0. So, $lim_{x\to x_0} f(x) = l_1$ and $lim_{x\to x_0} f(x) = l_2$ where $l_1 \neq l_2$. Therefore, we have:

$$\forall \varepsilon > 0,\ \exists \delta_1 > 0, \text{ such that } \forall x \in D:\ 0 < |x - x_0| < \delta \iff |f(x) - l_1| < \varepsilon$$

and

$$\forall \varepsilon > 0,\ \exists \delta_2 > 0, \text{ such that } \forall x \in D:\ 0 < |x - x_0| < \delta \iff |f(x) - l_2| < \varepsilon$$

Consider $\delta = \max\{\delta_1, \delta_2\}$, Then, we have both

$$\forall \varepsilon > 0,\ \exists \delta > 0, \text{ such that } \forall x \in D:\ 0 < |x - x_0| < \delta \iff |f(x) - l_1| < \varepsilon \text{ and } |f(x) - l_2| < \varepsilon$$

let $\varepsilon = \frac{l_1 - l_2}{4}$. So for $0 < |x - x_0| < \delta$, we have,

$$\begin{aligned} |l_1 - l_2| &= |l_1 - f(x) + f(x) - l_2| \\ &\leq |f(x) - l_1| + |f(x) - l_2| \\ &\leq 2\varepsilon \\ &= \frac{l_1 - l_2}{2} \end{aligned}$$

This is a contradiction. The assumption that $l_1 \neq l_2$ cannot be true. Therefore, the limit is unique.

3.7.1.1 Sequential limits

Theorem 3.2

Let $f : D \to \mathbb{R}$. Then, the following are equivalent:

1. $\lim_{x\to x_0} f(x) = l$ *and*
2. *for every sequence $\{x_n\}$ in D such that $x_n \to x_0$ and $x \neq x_0$, we have $f(x_n) \to l$.*

Proof($1. \implies 2.$):

Suppose $f(x) \to l$ as $x \to x_0$. This, according to the definition of the limit, is equivalent to saying that

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \forall x \in D : \ 0 < |x - x_0| < \delta \implies |f(x) - l| < \varepsilon$$

Let $\{x_n\}$ be a sequence in D such that $x_n \to x_0$. Then,

$$\forall \varepsilon_1 > 0, \exists n_0 \in \mathbb{N}, \text{ such that } \forall n \in \mathbb{N} : \ n \geq n_0 \implies |x_n - x_0| < \varepsilon_1$$

Choose $\varepsilon_1 = \delta$. Then, by combining the two previous implications, it is evident that we have

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \text{ such that } \forall n \in \mathbb{N} : \ n \geq n_0 \implies |f(x_n) - l| < \varepsilon$$

Thus,

$$\lim_{x\to x_0} f(x_n) = l$$

($2. \impliedby 1.$):

Suppose 2. holds, and assume for the sake of contradiction that 1 is false. Then,

$$\lim_{x\to x_0} f(x) \neq l \iff \exists \varepsilon_0 > 0, \forall \delta > 0 \text{ such that } \exists x \in D : 0 < |x - x_0| < \delta \ \text{ and } \ |f(x) - l| \geq \varepsilon.$$

Let's choose $\delta = \frac{1}{n}$, $n \in \mathbb{N}^*$. Then,

$$\forall n \in \mathbb{N}, \exists x_n \in D \ \text{ such that } 0 < |x_n - x_0| < \frac{1}{n} \text{ and } |f(x_n) - l| \geq \varepsilon$$

Thus, we have constructed a sequence $\{x_n\}$ such that $x_n \neq x$, $x_n \in]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}[$, then

$$\lim_{n\to\infty} x_n = x$$

Then, by 2,

$$|f(x_n) - l| \geq \varepsilon$$

Which is a contradiction.

Example 3.6

We use the previous theorem to show that a function does not have a limit. For example, consider the function $f(x) = \sin(\frac{1}{x})$. This function does not have a limit at the point $x_0 = 0$; indeed, there exists a sequence $\{x_n\}$ defined by

$$x_n = \frac{1}{\frac{\pi}{2} + n\pi}$$

Such that

$$\lim_{n\to\infty} x_n = 0 = x_0$$

But on the other hand, the sequence

$$f(x_n) = f\left(\frac{1}{\frac{\pi}{2} + n\pi}\right) = \sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n$$

does not have a limit as $n \to \infty$.

3.7.1.2 Limits and inequalities

Theorem 3.3

Let $D \subset \mathbb{R}$, and $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ are functions for which the limits of $f(x)$ and $g(x)$ exist as x approaches x_0, and

$$f(x) \leq g(x) \quad \forall x \in D$$

Then,

$$\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x)$$

Proof Let $l_1 = \lim_{x \to x_0} f(x)$ and $l_2 = \lim_{x \to x_0} g(x)$. Let $\{x_n\}$ be a sequence in D such that $x_n \to x_0$. Then, $\forall n \in \mathbb{N}$, $f(x_n) \leq g(x_n)$. Therefore,

$$l_1 = \lim_{n \to \infty} f(x_n) \leq \lim_{n \to \infty} g(x_n) = l_2.$$

Theorem 3.4 (Squeeze theorem)

Let $D \subset \mathbb{R}$, and $g : D \to \mathbb{R}$, $f : D \to \mathbb{R}$, $h : D \to \mathbb{R}$ are functions for which the limits of $g(x)$, $f(x)$ and $h(x)$ exist as x approaches x_0, the inequality

$$g(x) \leq f(x) \leq h(x) \quad \forall x \in D$$

holds, and

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = l$$

then it follows that

$$\lim_{x \to x_0} f(x) = l$$

Proof Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{whenever } 0 < |x - x_0| < \delta.$$

Since $\lim_{x \to x_0} g(x) = l$, by the definition of limits there exists $\delta_1 > 0$ such that

$$|g(x) - l| < \varepsilon \quad \text{for all } 0 < |x - x_0| < \delta_1.$$

That is,

$$l - \varepsilon < g(x) < l + \varepsilon \quad \text{for all } 0 < |x - x_0| < \delta_1. \tag{3.1}$$

Similarly, since $\lim_{x \to x_0} h(x) = l$, there exists $\delta_2 > 0$ such that

$$l - \varepsilon < h(x) < l + \varepsilon \quad \text{for all } 0 < |x - x_0| < \delta_2. \tag{3.2}$$

Moreover, since $g(x) \leq f(x) \leq h(x)$ holds in some open interval containing x_0, there exists

$\delta_3 > 0$ such that

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } 0 < |x - x_0| < \delta_3. \tag{3.3}$$

Now, let

$$\delta = \min(\delta_1, \delta_2, \delta_3).$$

Then, combining (3.1), (3.3), and (3.2), we obtain

$$l - \varepsilon < g(x) \leq f(x) \leq h(x) < l + \varepsilon \quad \text{for all } 0 < |x - x_0| < \delta.$$

Thus,

$$|f(x) - l| < \varepsilon \quad \text{for all } 0 < |x - x_0| < \delta.$$

Hence, by the definition of limits,

$$\lim_{x \to x_0} f(x) = l.$$

Example 3.7

Using the Squeeze theorem, show that

$$\lim_{x \to 0} x \sin\left(\tfrac{1}{x}\right) = 0$$

Since for all real x, we have

$$-1 \leq \sin\left(\tfrac{1}{x}\right) \leq 1,$$

it follows that

$$-x \leq x \sin\left(\tfrac{1}{x}\right) \leq x.$$

As $x \to 0$, both bounds satisfy

$$\lim_{x \to 0} -x = 0 \quad \text{and} \quad \lim_{x \to 0} x = 0.$$

Therefore, by the Squeeze Theorem, we conclude that

$$\lim_{x \to 0} x \sin\left(\tfrac{1}{x}\right) = 0.$$

Proposition 3.1 (Limits of absolute values)

Let $D \subset \mathbb{R}$, and Suppose $f : D \to \mathbb{R}$ is a function such that the limit of $f(x)$ exists as x goes to x_0. Then,

$$\lim_{x \to x_0} |f(x)| = |\lim_{x \to x_0} f(x)|$$

3.7.1.3 One-sided limits

Limits at a point x_0 introduced above are called two-sided because x approaches x_0 from both sides. Sometimes f is not defined on both sides of x_0 or we are only interested in function values on one side. That is why one-sided limits are defined. One-sided limits are limits that are approached from only one direction, either from the left or from the right. They are denoted as follows:

- The left-hand limit

$$\lim_{x \to x_0^-} f(x) \text{ or } \lim_{x \overset{<}{\to} x_0} f(x)$$

- The right-hand limit

$$\lim_{x\to x_0^+} f(x) \text{ or } \lim_{x \overset{>}{\to} x_0} f(x)$$

respectivelys is read as "the limit of $f(x)$ as x approaches x_0 from the left or from the right".

Definition 3.11

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$

- *We will say that f has a finite right-hand limit at x_0 if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in D \text{ such that } 0 < x - x_0 < \delta \implies |f(x) - l| < \varepsilon$$

- *We will say that f has a finite left-hand limit at x_0 if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in D \text{ such that } 0 < x_0 - x < \delta \implies |f(x) - l| < \varepsilon$$

Proposition 3.2

Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$. Then,

$$\lim_{x\to x_0} f(x) = l \iff \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = l.$$

Remark

To demonstrate that a function f does not have a limit at the point x_0, we can show that the right-hand limit is different from the left-hand limit.

Example 3.8

Consider the function:

$$f(x) = \begin{cases} x + 9 & \text{if } x > 0, \\ x - 9 & \text{if } x < 0. \end{cases}$$

Then,

$$\lim_{x\to 0^-} f(x) = -9 \text{ and } \lim_{x\to 0^+} f(x) = 9,$$

therefore, $\lim_{x\to 0} f(x)$ does not exist.

3.7.1.4 Infinite limits and limits at infinity of a function

Definition 3.12

Let $f : D \to \mathbb{R}$.

- *We say that the function f has the limit $+\infty$ as x approaches x_0, if and only if*

$$\lim_{x\to x_0} f(x) = +\infty \iff \forall A > 0,\ \exists \delta > 0,\ \forall x \in D :\ 0 < |x - x_0| < \delta \implies f(x) > A$$

- *We say that the function f has the limit $-\infty$ as x approaches x_0, if and only if*

$$\lim_{x\to x_0} f(x) = -\infty \iff \forall A > 0,\ \exists \delta > 0,\ \forall x \in D :\ 0 < |x - x_0| < \delta \implies f(x) < -A$$

- *We say that the function f has the limit $+\infty$ as x approaches $+\infty$, if and only if*

$$\lim_{x\to +\infty} f(x) = +\infty \iff \forall A > 0,\ \exists B > 0,\ \forall x \in D :\ x > B \implies f(x) > A$$

- *We say that the function f has the limit $+\infty$ as x approaches $+\infty$, if and only if*

$$\lim_{x\to+\infty} f(x) = -\infty \iff \forall A > 0, \exists B > 0, \forall x \in D : x > B \implies f(x) < -A$$

- *We say that the function f has the limit $+\infty$ as x approaches $+\infty$, if and only if*

$$\lim_{x\to-\infty} f(x) = +\infty \iff \forall A > 0, \exists B > 0, \forall x \in D : x < -B \implies f(x) > A$$

- *We say that the function f has the limit $+\infty$ as x approaches $-\infty$, if and only if*

$$\lim_{x\to-\infty} f(x) = -\infty \iff \forall A > 0, \exists B > 0, \forall x \in D : x < -B \implies f(x) < -A$$

3.7.1.5 Properties of limits and algebraic operations

Given the limits of two functions, f and g, we can determine, subject to certain conditions, the limits of their sum, difference, product, and quotient.

Proposition 3.3

Consider two real function $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ and assuming, limits $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$ at the point x_0 exist, then

- $\lim_{x\to x_0} (f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = l_1 + l_2$.
- $\lim_{x\to x_0} (f-g)(x) = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x) = l_1 - l_2$.
- $\lim_{x\to x_0} (fg)(x) = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x) = l_1.l_2$.
- $\lim_{x\to x_0} (\frac{f}{g})(x) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{l_1}{l_2}$ *and* $l_2 \neq 0$.
- *For* $\lambda \in \mathbb{R}$, $\lim_{x\to x_0} (\lambda f)(x) = \lambda \lim_{x\to x_0} f(x) = \lambda l_1$.

The statements are also true for one-sided limits.

Proposition 3.4

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined in the neighbourhood of a point x_0. If f is a bounded function and $\lim_{x\to x_0} f(x) = 0$, then

$$\lim_{x\to x_0} [f(x).g(x)] = 0$$

3.7.1.6 Indeterminate forms

Indeterminate forms are expressions that cannot be immediately determined or evaluated when applying the limit operation. Common indeterminate forms include:

$$\frac{\infty}{\infty};\ +\infty - \infty;\ \frac{0}{0};\ 0.\infty;\ 1^{\infty};\ \infty^{0};\ 0^{\infty}$$

3.7.1.7 Landau's O and o notation

Let f and g be two functions defined in a neighbourhood of a point $x_0 \in \mathbb{R}$.

Definition 3.13

We say that f is negligible compared to g as x approaches x_0, and we write $f = o(g)$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R} : 0 < |x - x_0| < \alpha \implies |f(x)| \leq \varepsilon |g(x)|$$

Remark

- $f = o(g) \iff \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$
- If $g(x) = 1, \forall x \in \mathbb{R}$, then $f = o(1) \iff \lim_{x \to x_0} f(x) = 0.$

Definition 3.14

We say that f is dominated by another function g as x approaches x_0, and we write $f = O(g)$ if

$$\exists K > 0, \exists \delta > 0, \forall x \in \mathbb{R} : 0 < |x - x_0| < \alpha \implies |f(x)| \leq K|g(x)|$$

Remark

- The symbols o and O are called Landau notations.
- If $g(x) = 1, \forall x \in \mathbb{R}$, then $f = O(1) \iff f$ is bounded on $V(x_0)$.

3.7.1.8 Equivalent functions

Equivalent functions are useful in simplifying expressions and evaluating limits, especially when dealing with more complex functions.

Definition 3.15

Let f and g be two functions neighbourhood neighbourhood of a point $x_0 \in \mathbb{R}$. We say that f is equivalent to g as x approaches x_0, and we denote it as $f \underset{x_0}{\sim} g$ if $f - g = o(f)$.

Remark

- $f \underset{x_0}{\sim} g \iff f - g = o(f) \iff f - g = o(g).$
- $f \underset{x_0}{\sim} g \iff \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$

Example 3.9

- $\sin x \underset{0}{\sim} x.$
- $\ln(x+1) \underset{0}{\sim} x$

Proposition 3.5

Let f, g, f_1 and g_1 be functions defined in a neighborhood x_0 such that $f \underset{x_0}{\sim} f_1$ and $g \underset{x_0}{\sim} g_1$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists, then $\lim_{x \to x_0} \frac{f_1(x)}{g_1(x)}$ also exists, and the two limits are equal.

Example 3.10

Using equivalent functions to calculate the following limit: $\lim_{x \to 0} \frac{\ln((\sin x)^2+1)}{\sin \frac{x}{2}}$
We have

$$\ln((\sin x)^2 + 1) \underset{0}{\sim} (\sin x)^2 \underset{0}{\sim} x^2 \text{ and } \sin \frac{x}{2} \underset{0}{\sim} \frac{x}{2}$$

Then

$$\frac{\ln((\sin x)^2+1)}{\sin\frac{x}{2}} \underset{0}{\sim} \frac{x^2}{\frac{x}{2}} \underset{0}{\sim} 2x$$

So,

$$\lim_{x\to 0}\frac{\ln((\sin x)^2+1)}{\sin\frac{x}{2}} = 0$$

3.7.2 Continuity

Using limit, we can establish another fundamental concept in mathematical analysis

3.7.2.1 Continuity of a function at a point

Definition 3.16

A function f is continuous at a point $x_0 \in \mathbb{R}$ if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

A function f is left-continuous or continuous from the left at a point $x_0 \in \mathbb{R}$ if

$$\lim_{x\to x_0^-} f(x) = f(x_0)$$

A function f is right-continuous or continuous from the right at a point $x_0 \in \mathbb{R}$ if

$$\lim_{x\to x_0^+} f(x) = f(x_0)$$

In particular, if

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = f(x_0)$$

then f is said to be continuous at $x = x_0$.

Example 3.11

- Let f be a real function defined by:

$$f(x) = \begin{cases} \cos^2(\pi x) & \text{if } x \leq 1, \\ 1 + \frac{\ln(x)}{x} & \text{if } x > 1. \end{cases}$$

We have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos^2(\pi x) = (-1)^2 = 1 = f(1);$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(1 + \frac{\ln x}{x}\right) = 1 = f(1);$$

Thus, f is continuous at $x = 1$ because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1);$$

- Let the function

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & -x < 0 \end{cases}$$

  This function is continuous at all x_0.

Remark

If the function f is not continuous at the point x_0, we say that f is discontinuous at x_0 and x_0 is a point of discontinuity of f.

Example 3.12

Let the function

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is not continuous at $x = 0$. So,

$$\lim_{x \to 0} f(x) = 1 \neq f(0)$$

3.7.2.2 Continuity of a function in an interval

Definition 3.17

A function $f(x)$ is said to be continuous on an interval $[a, b]$ if the following three conditions are satisfied:

- *$f(x)$ is defined on the interval $[a, b]$.*
- *$f(x)$ is continuous at every point in the interval $]a, b[$.*
- *f is right continuous at a point a and left continuous at a point b;*

$$\lim_{x \to x_0^+} f(x) = f(a) \text{ and } \lim_{x \to x_0^-} f(x) = f(b)$$

Example 3.13

The function $f(x) = \sqrt{4 - x^2}$ is defined when the expression under the square root is nonnegative:

$$4 - x^2 \geq 0 \implies -2 \leq x \leq 2.$$

- For every $x \in (-2, 2)$, the function is continuous because the square root function is continuous wherever it is defined:

$$\lim_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2}.$$

- And

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 = f(-2), \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0 = f(2),$$

  showing right continuity at $x = -2$ and left continuity at $x = 2$.

Therefore, $f(x)$ is continuous on the interval $[-2, 2]$.

Proposition 3.6

Polynomials, exponential and logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and the power function are continuous on their natural domains.

3.7.2.3 Continuous extension at a point

The concept of a "continuous extension at a point" refers to extending the domain of a function so that it becomes continuous at a specific point where it might not have been defined or continuous initially.

Definition 3.18

Let D be an interval and $f(x)$ be a function continuous for all $x \in D$ except at x_0 if $\lim_{x \to x_0} f(x)$ exists and is equal to a real number l. Then f can be extended by continuity to the function such that

$$\tilde{f}(x) = \begin{cases} f(x) & x \in D - \{x_0\} \\ l & x = x_0 \end{cases}$$

Remark

The function $\tilde{f}$ is continuous on interval D.

Example 3.14

Let $f(x) = \frac{\sin x}{x}$, the domain of f is $\mathbb{R}^*$. f is discontinuous at 0 because $f(0)$ is not defined. We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$, then the discontinuity is removable and redefine the function by

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

3.7.2.4 Uniform continuity

Definition 3.19

Let D be a nonempty subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is uniformly continuous on D if,

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x_1,\ x_2 \in D : |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

Example 3.15

The function $f(x) = x^2$ is uniformly continuous on the interval $]0, 1]$.
Given $\varepsilon > 0$ and let $x_1,\ x_2 \in]0, 1]$ then we have

$$0 < x_1 \leq 1 \text{ and } 0 < x_2 \leq 1 \implies 0 < x_1 + x_2 < 2$$

Or

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2|$$

Then

$$|f(x_1) - f(x_2)| \leq 2|x_1 - x_2|$$

So, it is enough to take $\delta = \frac{\varepsilon}{2} > 0$.

3.7.2.5 Lipschitz function

Definition 3.20

Let $D \subset \mathbb{R}$, a function $f : D \to \mathbb{R}$ be said to be Lipschitz continuous if there exists a constant K such that for all $x_1, x_2 \in D$, the following inequality holds:

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

Example 3.16

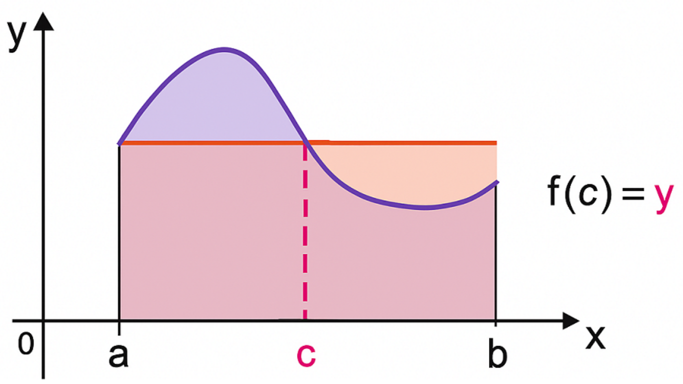
The function $f(x) = x^2$ is Lipschitz continuous on the interval $[0, b]$ with $b > 0$, but it does not satisfy a Lipschitz condition on the unbounded interval $[0, \infty)$.

Remark

A Lipschitz function on D is uniformly continuous on D.

Theorem 3.5 (The intermediate value theorem)

If a function f is continuous in a closed interval $[a, b]$ and if y is some number between $f(a)$ and $f(b)$ then there is a number $c \in [a, b]$ such that $f(c) = y$.



Proof Without loss of generality, suppose $f(a) < y < f(b)$. Define

$$A := \{x \in [a, b] : f(x) \leq y\}.$$

Since $f(a) < y$, we have $a \in A$, so A is nonempty. By definition, A is bounded above by b. Therefore, by the least upper bound axiom, A has a least upper bound, which we denote by $c = \sup A$.

We now show that $f(c) = y$.

First, note that $c \in [a, b]$ since $a \leq c \leq b$.

Let $\varepsilon > 0$ be arbitrary. Since f is continuous at c, there exists $\delta > 0$ such that for all $x \in [a, b]$ with $|x - c| < \delta$, we have

$$|f(x) - f(c)| < \varepsilon,$$

which is equivalent to

$$f(x) - \varepsilon < f(c) < f(x) + \varepsilon. \tag{1}$$

Because $c = \sup A$, the number $c - \delta$ is not an upper bound of A. Hence there exists $x_1 \in A$ with

$$c - \delta < x_1 \le c.$$

Since $|x_1 - c| < \delta$, applying the right-hand inequality in (1) gives

$$f(c) < f(x_1) + \varepsilon \le y + \varepsilon,$$

because $x_1 \in A$ implies $f(x_1) \le y$. Since ε was arbitrary, we conclude

$$f(c) \le y. \tag{2}$$

Moreover, $c < b$. Otherwise, if $c = b$, then $f(b) = f(c) \le y$, contradicting $y < f(b)$.

Now pick x_2 with $c < x_2 < b$ and $|x_2 - c| < \delta$. Since $x_2 > c = \sup A$, we have $x_2 \notin A$, which means $f(x_2) > y$. Applying the left-hand inequality in (1), we get

$$f(c) > f(x_2) - \varepsilon > y - \varepsilon.$$

Since ε was arbitrary, this implies

$$f(c) \ge y. \tag{3}$$

Combining (2) and (3), we obtain

$$f(c) = y.$$

Thus, there exists $c \in [a, b]$ with $f(c) = y$, as required.

Example 3.17

Show that equation $\cos x = x$ has a solution in the interval $]0, \frac{\pi}{2}[$. Let $f(x) = \cos x - x$, f be continuous on $[0, \frac{\pi}{2}]$ and satisfy $f(0) = 1 > 0$ and $f(\frac{\pi}{2}) = -\frac{\pi}{2} < 0$. According to the Mean Value Theorem, there exists at least one solution such that $f(c) = 0$. Therefore, c is a solution of equation $\cos(x) = x$

Proposition 3.7

If the functions f and g are continuous at a point x_0, then the functions $f + g$,, $f - g$ and $f.g$ are continuous at x_0. If, moreover, $g(x_0) \neq 0$ the function $\frac{f}{g}$ is continuous at x_0.

Proposition 3.8

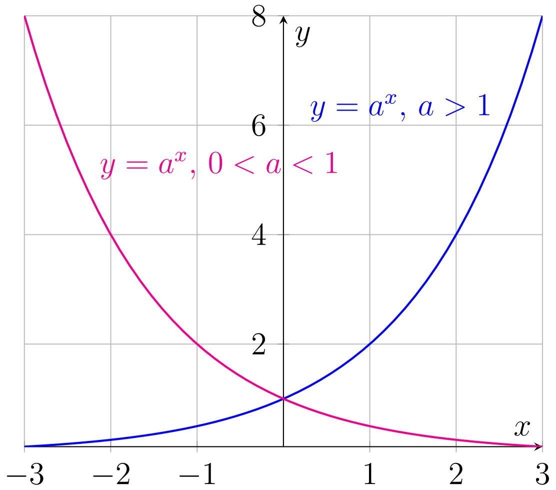
Let function f be continuous at the point x_0 and let function g be continuous at point $y_0 = f(x_0)$, then the composite function $f \circ g$ is continuous at the point x_0.

3.8 Elementary functions

This section focuses on the study of elementary functions which appear naturally in the solution of basic problems, especially physics issues. In this regard, we introduce the fundamental concepts of these functions and explore some of their key properties.

3.8.1 Exponential Function

Let $a \in \mathbb{R}$, $a > 0$. The function $f(x) = a^x$, $x \in \mathbb{R}$ is called the exponential function with base a.



3.8.1.1 Properties

- Domain: $(-\infty, +\infty)$.
- The function is increasing for $a > 1$, decreasing for $0 < a < 1$, and constant for $a = 1$.
- Let $a \in \mathbb{R}^+$, then for any $x, y \in \mathbb{R}$, the following laws of exponents hold:

$$a^{x+y} = a^x \cdot a^y, \quad a^{-x} = \frac{1}{a^x}, \quad a^{x-y} = \frac{a^x}{a^y}, \quad (a^x)^y = a^{xy}$$

Remark

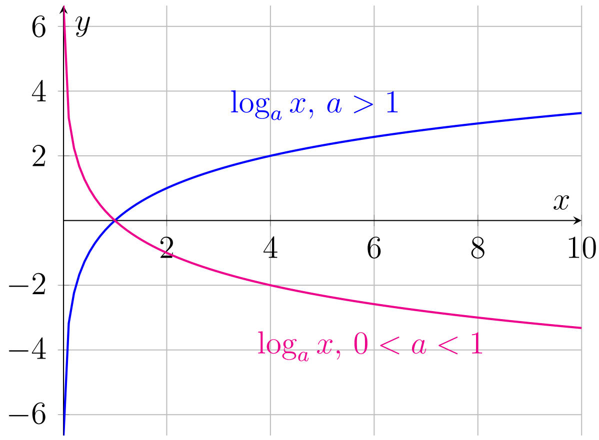
- The most important choice of base a is Euler's number $e \approx 2.718281828\ldots$ Later, we will see why this choice is so important and provide a definition of e. The function e^x is called the natural exponential function.
- $\forall x \in \mathbb{R}$, $e^x > 0$.
- The function e^x is strictly increasing.
- $\forall x, y \in \mathbb{R}$, $e^x = e^y \iff x = y$, and $e^x < e^y \iff x < y$.

3.8.1.2 Some Reference Limits

1. $\lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to+\infty} e^x = +\infty, \quad \lim_{x\to 0} \frac{e^x-1}{x} = 1.$
2. $\lim_{x\to+\infty} \frac{x^n}{e^x} = 0, \quad \lim_{x\to\infty} x^n e^{-x} = 0, \quad \forall n \in \mathbb{N}.$

3.8.2 Logarithmic Function

Consider the function $f(x) = a^x$, where $a > 0$, $a \neq 1$, and $x \in \mathbb{R}$. This function is strictly monotonic, and its inverse f^{-1} exists. It is called the logarithmic function with base a and is denoted $\log_a x$.



3.8.2.1 Properties

- Domain: $(0, +\infty)$.
- The function is increasing for $a > 1$ and decreasing for $0 < a < 1$.
- Let $a > 0$, $a \neq 1$, $x, y \in \mathbb{R}^+$. Then:

$$\log_a(xy) = \log_a x + \log_a y, \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^n) = n \log_a x, \quad \forall n \in \mathbb{N}$$

Remark

- The logarithm with base e is called the natural logarithm and is denoted by $\ln$.
- The natural logarithmic function $y = \ln x$ is the inverse of the exponential function $y = e^x$. That is,

$$\forall x > 0 : x = e^y \iff \ln x = y$$

- If $a = 10$, the logarithm is called the common logarithm, denoted by $\log$. It is used in chemistry. Base-2 logarithms are common in computer science.

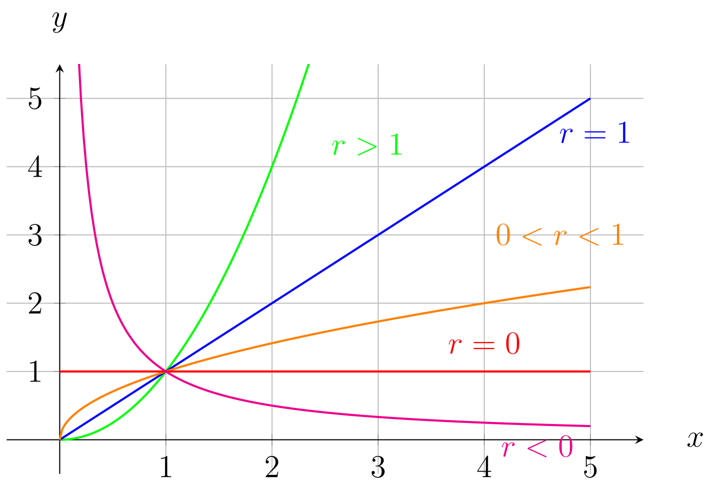
3.8.2.2 Some Reference Limits

1. $\lim\limits_{x\to 0^+} \ln x = -\infty, \quad \lim\limits_{x\to +\infty} \ln x = +\infty, \quad \lim\limits_{x\to 0} \frac{\ln(x+1)}{x} = 1.$
2. $\lim\limits_{x\to +\infty} \frac{\ln x}{x^n} = 0, \quad \lim\limits_{x\to 0^+} x^n \ln x = 0, \quad \forall n \in \mathbb{N}.$

3.8.3 Power Function

Let $r \in \mathbb{R}$. The function $f(x) = x^r$, $x > 0$ is the power function. It can be expressed using exponential and logarithmic functions:

$$x^r = (e^{\ln x})^r = e^{r \ln x}, \quad x > 0,\ r \in \mathbb{R}$$

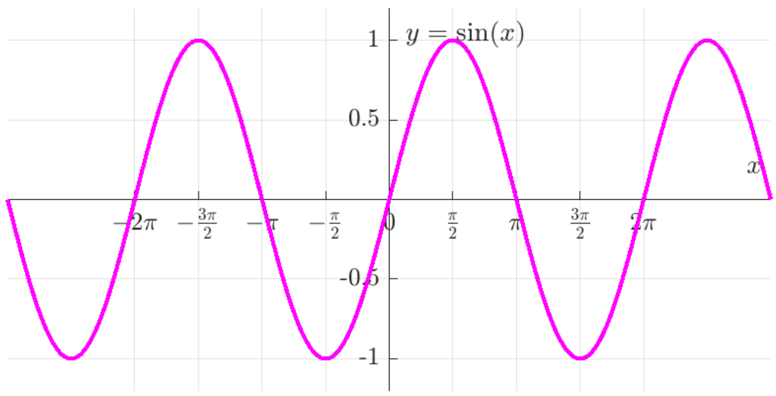


3.9 Trigonometric functions

This section is devoted to trigonometric functions, which play a central role in mathematics and its applications. They emerge naturally in the study of periodic phenomena, geometric problems, and especially in physics and engineering contexts. We introduce the basic definitions of these functions, discuss their fundamental properties.

3.9.1 Sine

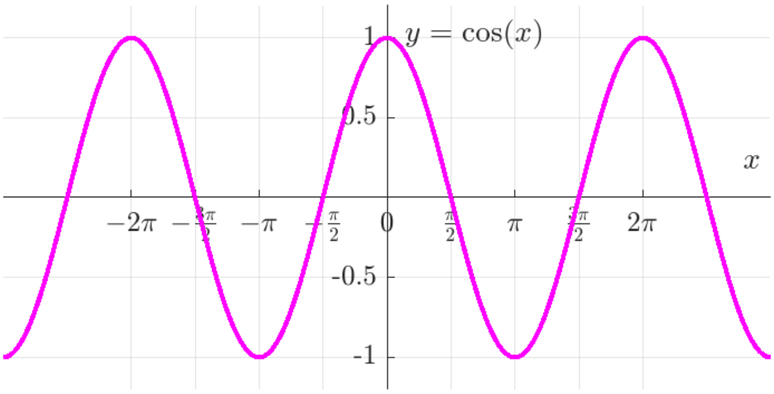
- Notation: $f : y = \sin x$.
- Domain: $\mathbb{R}$.
- It is odd, that is, $\sin(-x) = -\sin x$.
- It is periodic, its primitive period is 2π, that is, $\sin(x + 2k\pi) = \sin x,\ k \in \mathbb{Z}$.
- It increases on intervals $]-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi[$, $k \in \mathbb{Z}$, and decreases on intervals $]\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi[$, $k \in \mathbb{Z}$.
- $-1 \leq \sin x \leq 1$.



3.9.2 Cosine

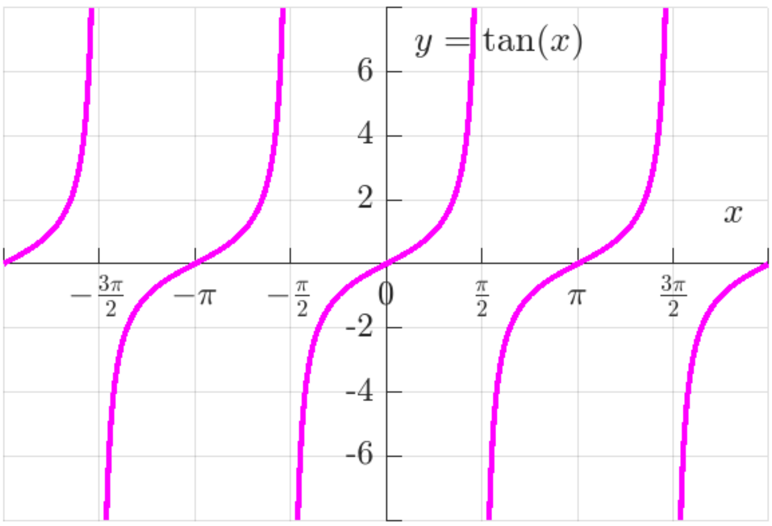
- Notation: $f : y = \cos x$.

- Domain: $\mathbb{R}$.
- It is even, that is, $\cos(-x) = \cos x$.
- It is periodic, its primitive period is 2π, that is, $\cos(x + 2k\pi) = \cos x,\ k \in \mathbb{Z}$.
- It increases on intervals $]-\pi + 2k\pi, 2k\pi[$, and decreases on intervals $]2k\pi, \pi + 2k\pi[$.
- $-1 \leq \cos x \leq 1$.



3.9.3 Tangent

- Notation: $f : y = \tan x$ defined by formula $\tan = \frac{\sin x}{\cos x}$ for $\cos x \neq 0$.
- Domain: $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi\right\},\ k \in \mathbb{Z}$.
- It is odd, that is, $\tan(-x) = -\tan x$.
- It is periodic, its primitive period is π, that is, $\tan(x + \pi) = \tan x,\ k \in \mathbb{Z}$.
- It increases on intervals $]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi[$, $k \in \mathbb{Z}$.

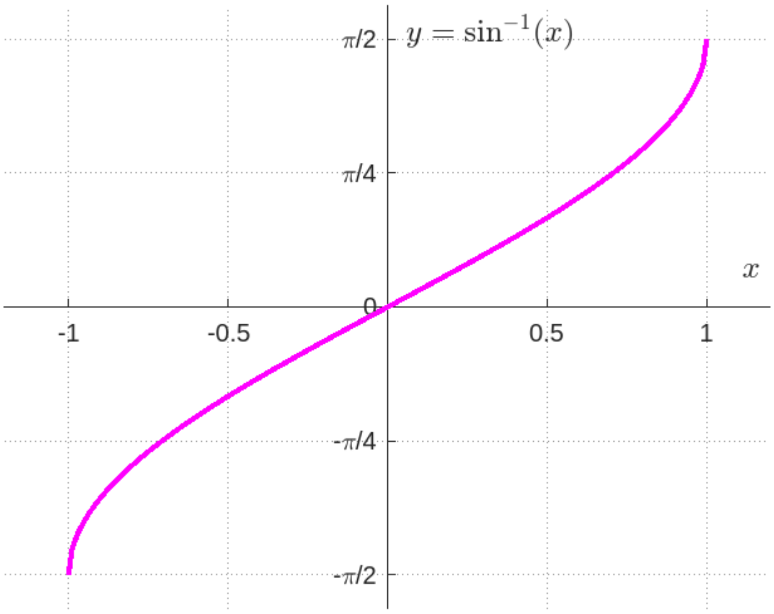


3.10 Inverse trigonometric functions

This section focuses on inverse trigonometric functions, which determine the angles corresponding to given trigonometric values. These functions are fundamental in mathematics, as they help us understand the relationship between angles and ratios. We will define each inverse function, explore their key properties, and study their graphs to visualize their behavior.

3.10.1 Inverse Sine

- Notation: $f^{-1} : y = \arcsin x$ or $f : y = \sin^{-1} x$.
- Domain: $]-1, 1[$.
- It is odd, that is, $\arcsin(-x) = -\arcsin x$.
- It is not periodic.
- It is increasing.

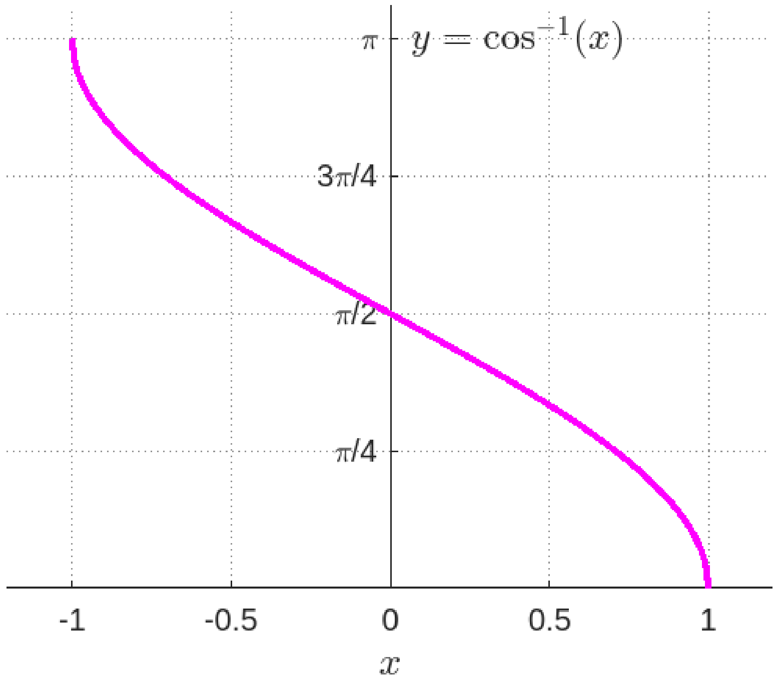


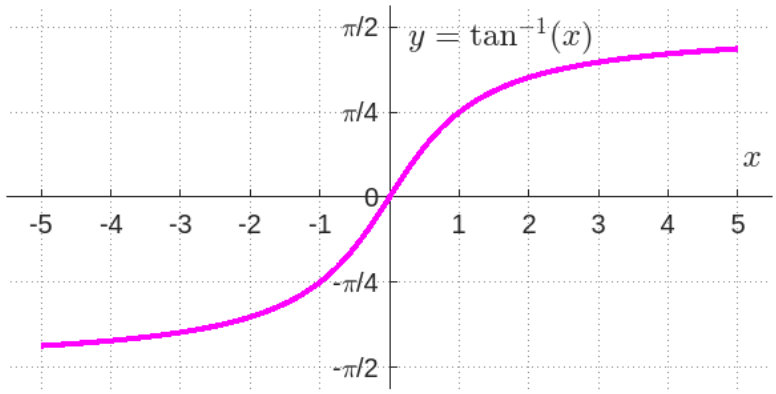
3.10.2 Inverse Cosine

- Notation: $f^{-1} : y = \arccos x$ or $f : y = \cos^{-1} x$.
- Domain: $]-1, 1[$.
- It is neither odd nor even.
- It is not periodic.
- It is decreasing.

3.10.3 Inverse Tangent

- Notation: $f^{-1} : y = \arctan x$ or $f : y = \tan^{-1} x$.
- Domain: $\mathbb{R}$.
- It is odd, that is, $\arctan(-x) = -\arctan x$.
- It is not periodic.
- It is increasing.





3.11 Hyperbolic functions

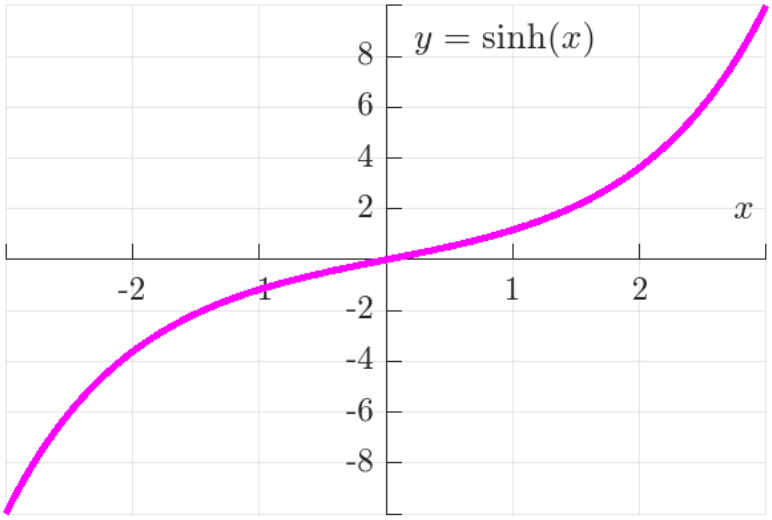
Three functions are included among the hyperbolic functions: hyperbolic sine, hyperbolic cosine, and hyperbolic tangent. These are widely utilized in technical applications across diverse fields, including physics, engineering, and mathematics. In the following discussion, we will explore their fundamental properties and graphs.

3.11.1 Hyperbolic sine

The hyperbolic sine function is defined by the formula

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

- Domain: $\mathbb{R}$.
- It is odd, i.e., $\sinh(-x) = -\sinh(x)$.
- It is not periodic.
- It is increasing.

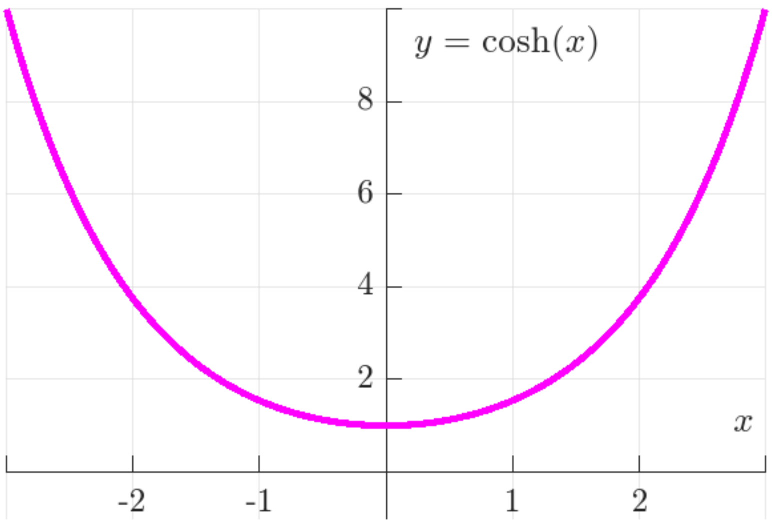


3.11.2 Hyperbolic Cosine

The hyperbolic cosine function is defined by the formula

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

- Domain: $\mathbb{R}$.
- It is even, i.e., $\cosh(-x) = \cosh(x)$.
- It is not periodic.
- It increases on the interval $[0, +\infty[$ and decreases on the interval $]-\infty, 0]$.



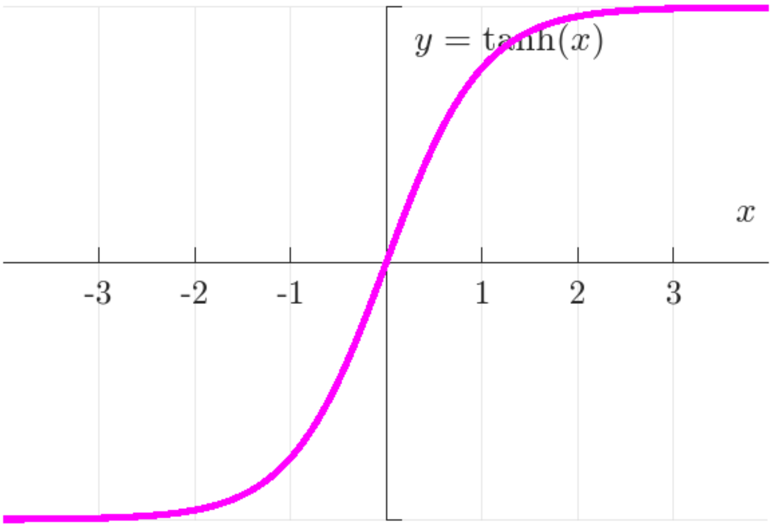
3.11.3 Hyperbolic Tangent

The hyperbolic tangent function is defined by the formula

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- Domain: $\mathbb{R}$.
- It is odd, i.e., $\tanh(-x) = -\tanh(x)$.

- It is not periodic.
- It is increasing.



3.12 Derivative

This section begins with the definition of the derivative.

Definition 3.21

Let f be a real-valued function. We say that f is differentiable at x_0 in its domain D_f if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = l \in \mathbb{R}$$

Exists. In this case, the value l is referred to as the derivative of f at x_0. The derivative of f at x_0, if it exists, is denoted by $f'(x_0)$, read as f prime of x_0.
We can express the analogous definition as follows:
f is differentiable at x if and only if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \in \mathbb{R}.$$

Example 3.18

The function $f : x \mapsto \sqrt{x}$ is differentiable at $x_0 = 1$. To demonstrate this, consider the limit

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1}.$$

Simplifying the expression, we get

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$$

Remark

A function defined on an open interval I from $\mathbb{R}$ to $\mathbb{R}$ is said to be differentiable on I if it is differentiable at every point in I.

Definition 3.22 (Left and right-differentiable)

1. *f is left-differentiable at a point x_0 if the limit*
$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = f'_L(x_0)$$
Exists.
2. *f is right-differentiable at a point x_0 if the limit*
$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = f'_R(x_0)$$
Exists.

These definitions describe left differentiability and right differentiability at a point x_0, denoted by $f'_L(x_0)$ and $f'_R(x_0)$ respectively.

Proposition 3.9

A function f is differentiable at a point x_0 if and only if f is left-differentiable and right-differentiable at this point, i.e.,
$$f'_L(x_0) = f'_R(x_0).$$

Proposition 3.10

If f is a differentiable function at x_0, then f is continuous at x_0.

Remark

Every differentiable function is continuous, but the converse is not necessarily true.

Example 3.19

The function $x \mapsto |x|$ is continuous but not differentiable at $x = 0$.

3.12.1 Derivatives of elementary functions

$f(x)$	$f'(x)$
ax	a
x^n	nx^{n-1}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
e^x	e^x
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$
$\sin^2 x$	$2\sin x\cos x$
$\cos^2 x$	$-2\sin x\cos x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

Proposition 3.11

Suppose that functions f and g have derivatives at the point $x_0 \in \mathbb{R}$. Then the functions $f+g$, fg, $\frac{f}{g}$ (if $g(x_0) \neq 0$), and cf, where $c \in \mathbb{R}$ is a constant, have derivatives at x_0, and the following formulas hold:

1. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$ *(sum rule),*
2. $(f-g)'(x_0) = f'(x_0) + g'(x_0)$ *(difference rule),*
3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ *(product rule),*
4. $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}$ *for* $g(x_0) \neq 0$ *(quotient rule),*
5. $(cf)'(x_0) = cf'(x_0)$ *(constant multiple rule).*

Theorem 3.6 (Derivative of inverse function)

Assume that the function $f : I \to \mathbb{R}$, where $x = f(y)$, is continuous and strictly monotonic on the interval I. Let y_0 be an interior point of I, and assume that $f'(y_0)$ exists. Then, the inverse function f^{-1} has a derivative at the point $x_0 = f(y_0)$ given by the formula:

$$(f^{-1})'(x_0) = \frac{1}{f'(y_0)}.$$

Proof By the definition of the inverse function, we have

$$f(f^{-1}(x)) = x, \quad \text{for all } x \in f(I).$$

Differentiating both sides with respect to x, and applying the chain rule, we get

$$\frac{d}{dx}\left[f(f^{-1}(x))\right] = f'\big(f^{-1}(x)\big) \cdot (f^{-1})'(x) = \frac{d}{dx}[x] = 1.$$

Evaluating this at the point $x_0 = f(y_0)$, we obtain

$$f'(y_0) \cdot (f^{-1})'(x_0) = 1.$$

Solving for $(f^{-1})'(x_0)$, we conclude

$$(f^{-1})'(x_0) = \frac{1}{f'(y_0)}.$$

Example 3.20

The inverse of the function $f(x) = x^2$ with reduced domain $[0, \infty)$ is

$$f^{-1}(x) = \sqrt{x}.$$

We have

$$f'(x) = 2x, \quad \text{so that} \quad f'(f^{-1}(x)) = 2\sqrt{x}.$$

Using the theorem of derivative of inverse function, we obtain

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2\sqrt{x}}.$$

By the power rule,

$$(f^{-1})'(x) = \frac{d}{dx}\sqrt{x} = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Thus, the result is verified.

Example 3.21

Consider the function $f(x) = e^x$, defined for all real x. Its inverse is

$$f^{-1}(x) = \ln(x), \quad x > 0.$$

First, we compute

$$f'(x) = e^x.$$

Substituting $f^{-1}(x) = \ln(x)$ into this derivative gives

$$f'(f^{-1}(x)) = f'(\ln(x)) = e^{\ln(x)} = x.$$

Therefore, by the formula for the derivative of the inverse function,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{x}.$$

Direct differentiation confirms this result:

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Proposition 3.12 (Derivative of composite function)

Consider the composite function $F = f \circ g$. Assume that g has a derivative at the point x_0, and f has a derivative at the point $u_0 = g(x_0)$. Then the composite function F has a derivative at *the point x_0, and the following formula, known as the chain rule, holds:*

$$F'(x_0) = (f \circ g)'(x_0) = f'(u_0) \cdot g'(x_0) = f'\big|_{u=u_0}(g(x_0)) \cdot g'(x_0).$$

Now, we present the derivative of some composite function. If f is a differentiable function and α is any constant, then:

- $f^\alpha = \alpha f' f^{\alpha-1}$, where f is strictly positive.
- $(\sqrt{f})' = \frac{f'}{2\sqrt{f}}$, where f is strictly positive.
- $(e^f)' = f' e^f$.
- $(\ln f)' = \frac{f'}{f}$.
- $(\sin f)' = f' \cos f$.
- $(\cos f)' = -f' \sin f$.
- $(\tan f)' = \frac{f'}{\cos^2 f}$.

3.12.2 Higher order derivatives

In the previous section, we explained that if the function f has a derivative at every point we obtain a new function f' and this new function can have a derivative at a point x_0, denoted as $f''(x_0)$, if it exists. This number is called the second derivative of f at point x_0, and is denoted $f''(x_0)$. Therefore,

$$f''(x_0) = (f')'(x_0).$$

If f'' exists at every point we get a new function f''. This function can be differentiated at a point x_0 (provided it is possible), and we obtain the third derivative of f at point x_0, denoted $f'''(x_0)$. The process continues, and for $n = 4$, a dash is not used as the symbol of the derivative since such notation would be difficult to read. We denote $f', f'', f''', f^{(4)}, f^{(5)}$, etc. Round brackets cannot be omitted: $f^{(4)}$ is the fourth power of f, while $f^{(4)}(x_0)$ is the fourth derivative of f at point x_0. Moreover, it is useful to denote $f^{(0)} = f$.

Definition 3.23

Let $n \in \mathbb{N}$. The n-th derivative (or n-th order derivative) of the function f at the point x_0 is denoted as $f^{(n)}(x_0)$ and is defined recursively by the equality

$$f^{(n)}(x_0) = \left(f^{(n-1)}\right)'(x_0).$$

Higher-order derivatives (third, fourth, etc.) appear in many important applications such as Taylor series expansions, wave equations, oscillations, and control theory. They allow us to describe not only the rate of change of a function but also its concavity, curvature, and general dynamical behavior.

Example 3.22

Find

1. the fourth order derivative of $f(x) = e^{4x}$ at $x = 9$.
2. the fifth order derivative of $f(x) = \sin(4x + 9)$ at $x = 5$.

Solution:

1. For $f(x) = e^{4x}$ we compute:

$$f'(x) = 4e^{4x}, \quad f''(x) = 16e^{4x}, \quad f^{(3)}(x) = 64e^{4x}, \quad f^{(4)}(x) = 256e^{4x}.$$

   Hence,

$$f^{(4)}(9) = 256e^{36}.$$

2. For $f(x) = \sin(4x + 9)$ we compute:

$$f'(x) = 4\cos(4x + 9), \quad f''(x) = -16\sin(4x + 9), \quad f^{(3)}(x) = -64\cos(4x + 9),$$
$$f^{(4)}(x) = 256\sin(4x + 9), \quad f^{(5)}(x) = 1024\cos(4x + 9).$$

   Therefore,

$$f^{(5)}(5) = 1024\cos(29).$$

3.12.3 Derivative Recurrence Relations

In many cases, derivatives of elementary functions follow simple recurrence patterns. For example, the trigonometric functions $\sin x$ and $\cos x$ satisfy:

$$\frac{d^n}{dx^n}\sin x = \sin\left(x + n\frac{\pi}{2}\right), \qquad \frac{d^n}{dx^n}\cos x = \cos\left(x + n\frac{\pi}{2}\right).$$

This means that each differentiation corresponds to a phase shift of $\frac{\pi}{2}$. For instance:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d^2}{dx^2}\sin x = -\sin x, \quad \frac{d^3}{dx^3}\sin x = -\cos x, \quad \frac{d^4}{dx^4}\sin x = \sin x.$$

Example 3.23

Consider $f(x) = e^{ax}$. Its n-th derivative is

$$f^{(n)}(x) = a^n e^{ax},$$

which shows that exponential functions reproduce themselves under differentiation.

Similarly, for $f(x) = \sin(bx)$,

$$f^{(n)}(x) = b^n \sin\left(bx + n\frac{\pi}{2}\right),$$

and for $f(x) = \cos(bx)$,

$$f^{(n)}(x) = b^n \cos\left(bx + n\frac{\pi}{2}\right).$$

These recurrence relationships simplify the computation of higher-order derivatives and are widely used in solving differential equations and series expansions.

3.12.4 Applications of derivatives

Theorem 3.7 (Rolle's Theorem)

Suppose the function f has the following properties:

- *It is continuous on the closed bounded interval $[a, b]$,*
- *It has a derivative on the open interval (a, b),*
- $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof We distinguish two cases.

Case 1: If f is constant on $[a, b]$, then any $c \in (a, b)$ works, since the derivative of a constant function is zero.

Case 2: Suppose f is not constant. Then there exists some $x_0 \in [a, b]$ with $f(x_0) \neq f(a)$. Assume, for instance, that $f(x_0) > f(a)$.

Since f is continuous on the closed and bounded interval $[a, b]$, by the Extreme Value Theorem it attains a maximum at some point $c \in [a, b]$. Moreover,

$$f(c) \geq f(x_0) > f(a).$$

Thus, $c \neq a$. But since $f(a) = f(b)$, we also cannot have $c = b$. Therefore $c \in (a, b)$.

At this point c, the function f has a local maximum and is differentiable, so by Fermat's Theorem,

$$f'(c) = 0.$$

Theorem 3.8 (Lagrange's Mean Value Theorem)

Let a, b be two real numbers with $a < b$. Suppose function f has the following properties

- *f is continuous on $[a, b]$.*
- *f is differentiable on (a, b).*

Then, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof Consider the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Since f is continuous on $[a, b]$ and differentiable on (a, b), and the linear function $(x - a)$ is everywhere continuous and differentiable, it follows that g is continuous on $[a, b]$ and differentiable on (a, b).

Evaluate g at the endpoints:

$$g(a) = f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = f(a),$$

and

$$g(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - (f(b) - f(a)) = f(a).$$

Hence $g(a) = g(b)$.

By Rolle's theorem applied to g, there exists $c \in (a, b)$ such that $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so at $x = c$ we get

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which implies

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Now we will describe a method that helps us in many cases where we evaluate the limit of a quotient $\frac{f(x)}{g(x)}$.

Theorem 3.9 (L'Hôpital's Rule)

Let $f(x)$ and $g(x)$ be differentiable on an interval I containing a, and assume that $g'(x) \neq 0$ on I for $x \neq a$. Suppose that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{or} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

Then, as long as the limits exist, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The proof of L'Hôpital's Rule makes use of the above generalization of the Mean Value theorem.

Example 3.24

Use L'Hôpital's Rule to calculate the following limit:

1. $\lim_{x \to 0} \frac{x \cos x}{x + \arcsin x}$
2. $\lim_{x \to \infty} \frac{4x+9}{9x^2-4}$

Solution:

1. The given limit is of the indeterminate form $\frac{0}{0}$ as $x \to 0$. Therefore, we can apply L'Hôpital's Rule.
   Let
   $$f(x) = x \cos x \quad \text{and} \quad g(x) = x + \arcsin x.$$
   Differentiating $f(x)$ and $g(x)$, we obtain:
   $$f'(x) = \cos x - x \sin x, \quad g'(x) = 1 + \frac{1}{\sqrt{1 - x^2}}.$$
   Using L'Hôpital's Rule, the limit becomes:
   $$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x - x \sin x}{1 + \frac{1}{\sqrt{1-x^2}}} = \frac{1}{2}.$$
2. Note that the limit is of the indeterminate form $\frac{\infty}{\infty}$ as $x \to \infty$. Therefore, by L'Hôpital's Rule we proceed as follows:
   $$f(x) = 4x + 9, \qquad g(x) = 9x^2 - 4.$$

Differentiating,

$$f'(x) = 4, \qquad g'(x) = 18x.$$

Hence,

$$\lim_{x\to\infty} \frac{4x+9}{9x^2-4} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{4}{18x}.$$

Since $\frac{1}{x} \to 0$ as $x \to \infty$, we conclude

$$\lim_{x\to\infty} \frac{4x+9}{9x^2-4} = 0.$$

Chapter 3 Exercises

Exercise 1

- Calculate the following limits if they exist

$$(1)\ \lim_{x\to\infty} \left(\frac{x-3}{x+3}\right)^x, \qquad (2)\ \lim_{x\to\infty} e^{sinx-x}, \qquad (3)\ \lim_{x\to 0} \frac{x}{|x|},$$

$$(4)\ \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} \qquad (5)\ \lim_{x\to\pi} \frac{\sin x}{x-\pi} \qquad (6)\ \lim_{x\to 0} \frac{x-\sin(2x)}{x+\sin(3x)}$$

- Using the definition of the limit, show that

$$(1)\ \lim_{x\to\infty} \frac{4x-9}{9x+4} = \frac{4}{9}, \qquad (2)\ \lim_{x\to-\infty} x^2 = +\infty, \qquad (3)\ \lim_{\substack{x\to-4 \\ x>-4}} \frac{9}{4+x} = +\infty,$$

Exercise 2

Consider the two functions f and g defined on $\mathbb{R}$ by:

$$f(x) = \begin{cases} \frac{x}{1+e^{\frac{1}{x}}} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad g(x) = \begin{cases} xe^{\frac{1}{x}} & x < 0 \\ 0 & x = 0 \\ x^2 \ln(1+\frac{1}{x}) & x > 0 \end{cases}$$

Study the continuity of f and g over their domains of definition.

Exercise 3

Study the extension by continuity of the following functions

$$(1)\ xe^{\arctan \frac{1}{x^2}}, \qquad (2)\ \cos \frac{1}{x}, \qquad (3)\ \frac{1-\cos x}{x(3-x)\tan x}$$

Exercise 4

Prove that

- $\forall x \in [-1,1] : \arcsin x + \arccos x = \frac{\pi}{2}$.
- $\forall x \in [-1,1] :\ \sin(\arccos x) = \sqrt{1-x^2}$.

Exercise 5

Study the differentiability of the functions f at a point
$x_0 = -1,\ x_0 = 1$ and for the functions g at a point $x_0 = 0$,

$$f(x) = \begin{cases} \arctan x & |x| \neq 1 \\ \frac{\pi}{4} sing(x) + \frac{x-1}{2} & |x| > 1 \end{cases},\ g(x) = \begin{cases} 2 + x\ln x & x > 0 \\ 1 + e^{-x} & x \leq 0 \end{cases}$$

Exercise 6

Use the L'Hopital's rule to find the following limit:

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}, \quad \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$$

Exercise 7

- Find the formula for the derivative of $arctanx$ starting from

$$\tan(\arctan x) = x$$

- Similarly, find the formula for the derivative of $arccotx$.
- Verify that

$$\frac{d}{dx} arccotx + \frac{d}{dx}\arctan x = 0$$

Chapter 4 Approximation of functions with polynomials

In this chapter, we will learn how functions can be approximated by polynomials. Taylor's theorem provides a way to replace complicated functions with simpler ones that are easier to work with. These polynomial approximations play a central role in analysis and applications. The main focus of this chapter will be on understanding these approximations and how they can be expressed using polynomials. After studying this chapter, you should be able to:

- Use Taylor's polynomial to find approximate values of functions.
- Use Taylor's polynomial to write series expansions of functions.
- Apply Taylor's formula with both the Lagrange remainder and the Young remainder.
- Understand and use the Taylor–Maclaurin–Young formula.
- Perform finite expansions of functions at zero and at any given point.
- Apply finite expansions to solve practical problems.

4.1 Taylor polynomial

For a function f that is continuous on the interval $[a, b]$ and differentiable at $x_0 \in]a, b[$, the following limit holds:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

from which we can write:

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \varepsilon(x)$$

with $\lim_{x \to x_0} \varepsilon(x) = 0$. Consequently, in the neighborhood of x_0, the function f can be expressed as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon(x)(x - x_0)$$

where, ε is a function such that $\lim_{x \to x_0} \varepsilon(x) = 0$. We say that f can be approximated by the first-degree polynomial T:

$$T(x) = f(x_0) + f'(x_0)(x - x_0)$$

Introducing an error term $R(x) = \varepsilon(x)(x - x_0) = o(x - x_0)$, which approaches 0 as x tends to x_0. More generally, we have the function f can be efficiently approximated near a point x_0 through the Taylor polynomial of degree n using additional derivatives $f'(x_0), f''(x_0), \ldots, f^{(n)}(x_0)$. This approximation is expressed as follows:

$$f(x) = T_n(x) + R_n(x)$$

Here, $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ represents a polynomial of degree n in $(x - x_0)$, while $R_n(x)$ denotes the error associated with this approximation, commonly referred to as the remainder of order n. The approximations of the function f by the Taylor polynomial of degree n is expressed as follows:

$$f(x) = f(x_0) + (x - x_0)\frac{f'(x_0)}{1!} + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + R_n(x)$$

In this context, $R_n(x)$ signifies the remainder of order n. To find the Taylor polynomials T_n of the function generated by $f(x) = \ln x$ at $x_0 = 1$ for $n = 1, 2, 3$, we need to evaluate the first three derivatives of f and find their values at 1:

$$\begin{aligned} f(x) &= \ln x, & f(1) &= 0, \\ f'(x) &= \frac{1}{x}, & f'(1) &= 1, \\ f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1, \\ f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2. \end{aligned}$$

Therefore,

- $T_1(x) = f(x_0) + f'(x_0)(x - x_0) = 0 + 1(x - 1) = x - 1$,
- $T_2(x) = T_1(x) + \frac{f''(x_0)}{2!}(x - x_0)^2 = (x - 1) - \frac{1}{2}(x - 1)^2$,
- $T_3(x) = T_2(x) + \frac{f'''(x_0)}{3!}(x - x_0)^3 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$.

The graphs of Taylor polynomials are

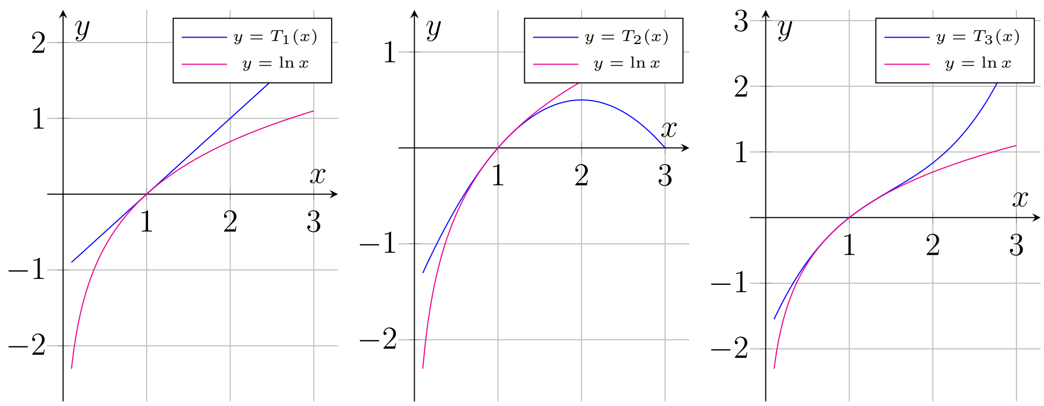


Figure 4.1: Taylor polynomials generated by $\ln x$ at $x_0 = 1$

4.2 Taylor formula with Lagrange remainder

Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in C^\infty([a, b])$, and $f^{(n)}$ is differentiable on $]a, b[$. Suppose $x_0 \in [a, b]$, then:

$$f(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

where c between x and x_0. The term $\frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$ is called the Lagrange remainder.

4.3 Taylor formula with Young's remainder

Let $f : [a,b] \to \mathbb{R}$ be a function, and let $x_0 \in [a,b]$ be such that $f^{(n)}(x_0)$ exists. Then:

$$f(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + o((x-x_0)^n)$$

where $R_n(x) = o((x-x_0)^n)$ is such that $\lim_{x\to x_0} \frac{R_n(x)}{(x-x_0)^n} = 0$. There exists a second expression for this formula by setting $\frac{R_n(x)}{(x-x_0)^n} = \varepsilon(x)$. Therefore, $R_n(x) = (x-x_0)^n \varepsilon(x)$, and consequently, we have:

$$f(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + (x-x_0)^n \varepsilon(x), \text{ with } \lim_{x\to x_0} \varepsilon(x) = 0$$

4.4 Taylor-Maclaurin-Young formula

If $x_0 = 0$, then we have:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + o(x^n)$$

or alternatively:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + x^n \varepsilon(x), \text{ where } \lim_{x\to 0} \varepsilon(x) = 0.$$

4.5 Finite expansions at zero

Let f be a real-valued function. We say that f has a finite expansion at zero if there exist real numbers $a_0, a_1, \ldots, a_n$ and a real-valued function ε such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + x^{(n)}\varepsilon(x),$$

where $\lim_{x\to 0} x^{(n)}(x) = 0$. Then f is represented by the polynomial approximation of degree n, denoted by $P_n(x)$ for x near zero. This approximation, referred to as the main term of the finite expansion at zero, is given by

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

and the remainder term is

$$x^{(n)}\varepsilon(x) = O(x^n)$$

Then

$$f(x) = P_n(x) + O(x^n)$$

4.5.1 Finite expansions of some elementary functions

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + O(x^n)$$

$$(1+x)^\lambda = 1 + \lambda x + \frac{\lambda(\lambda-1)x^2}{2!} + \ldots + \frac{\lambda(\lambda-1)\ldots(\lambda-(n-1))x^n}{n!} + O(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots + (-1)^n x^n + O(x^n)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + O(x^n)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+1})$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+1})$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n} + O(x^n)$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots - x^n + O(x^n)$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots - \frac{x^{2n+1}}{2n+1} + O(x^{2n+1})$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + O(x^{2n})$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+1})$$

4.5.2 Properties

- If a function f has a finite expansion at zero, that expansion is unique.
- Consider the finite expansions at zero of f and g:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + O(x^n)$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + O(x^n)$$

  - The finite expansion at zero of the sum $f + g$ is:

$$(f+g)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n + O(x^n)$$

  - The finite expansion at zero of the product $f \cdot g$ is obtained by multiplying the functions and retaining only the monomials of degree less than n in the resulting product:

$$(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n)(b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n)$$

  - The finite expansion at zero of the quotient $\frac{f}{g}$ is obtained through euclidean division of $(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n)$ by $(b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n)$, ordering the terms in increasing powers.
  - If g can be expanded at zero of degree n and if f can be expanded at $g(0)$ of degree n such that $g(0) = 0$, then the composite function $(f \circ g)$ can be expanded at zero of degree n by substituting the finite expansion of g into the finite expansion of f and retaining only the monomials of degree less than or equal to n.

Example 4.1

1. Find the finite expansion at zero of $f(x) = \cosh x$ (degree 4)
   Let $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$, we have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^4)$$

   and

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^4)$$

   then,

$$\begin{aligned}\cosh x &= \frac{1}{2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) + \frac{1}{2}\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}\right) + O(x^4) \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^4).\end{aligned}$$

2. Find the finite expansion at zero of $f(x) = \cos x \sin x$ (degree 5).
   We have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^5)$$

   and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^5)$$

then,

$$f(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) + O(x^5)$$
$$= \left(1 - \frac{x^2}{2}\right) + \left(x - \frac{x^3}{6}\right) + \frac{x^5}{120} + O(x^5)$$
$$= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 + O(x^5).$$

3. Find the finite expansion at zero of $f(x) = \frac{\ln(1+x)}{\sin x}$ (degree 3).
   Note that the function f can be expanded at zero up degree 3 if the finite expansions at zero of $\ln(1+x)$ and $\sin x$ are given (degree 4). Since

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^4)$$

and

$$\sin x = x - \frac{x^3}{3!} + O(x^4)$$

then,

$$f(x) = \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^4)}{x - \frac{x^3}{3!} + O(x^4)}$$
$$= \frac{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + O(x^3)}{1 - \frac{x^2}{3!} + O(x^3)}$$
$$= 1 - \frac{x}{2} + \frac{x^2}{6} - \frac{x^3}{12} + O(x^3).$$

4. Find the finite expansion at zero of $f(x) = e^{\cos x}$, (degree 3).
   If $g(x) = \cos x$, note that $g(0) = 1 \neq 0$. We have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^3)$$

and

$$\cos x = 1 - \frac{x^2}{2!} + O(x^3)$$

So if $g(x) = \cos x - 1 = -\frac{x^2}{2!} + O(x^3)$, in this case $g(0) = 0$. Put $X = -\frac{x^2}{2!} + O(x^3)$, so $x = 0 \implies X = 0$

$$e^{1+X} = ee^X = e(1 + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + O(x^3))$$

Now, replace X with a specific value we get

$$f(x) = e(1 - \frac{x^2}{2!} + O(x^3))$$
$$= e - e\frac{x^2}{2} + O(x^3)$$

4.6 Finite expansions at a point

Finite expansions of a function $f(x)$ at the point $x_0 = 0$ can be expressed as

$$f(x) = a_0 + a_1X + a_2X^2 + \ldots + a_nX^n + O(X^n), \quad \lim_{X\to 0} O(X^n) = 0.$$

Let $x = x_0 + h$ or $h = x - x_0$, then we have

$$f(x) = f(x_0 + h) = g(h)$$

if the function g admits a finite expansion at zero up degree n. Consequently,

$$g(h) = a_0 + a_1 h + a_2 h^2 + \ldots + a_n h^n + O(h^n), \quad \lim_{h \to 0} O(h^n) = 0.$$

Now, substituting h with $x - x_0$, the expression becomes

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n + O((x - x_0)^n).$$

Example 4.2

Find the finite expansion at 9 of $f(x) = e^x$ of degree 3.

Taking $x = 9 + h$ implies $h = x - 9$. The degree 3 expansion can be expressed as:

$$\begin{aligned} f(x) &= f(9 + h) \\ &= e^{9+h} \\ &= e^9 \cdot e^h \\ &= e^9 \cdot \left(1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^3)\right). \end{aligned}$$

Now, substituting h with $x - 9$, we get

$$= e^9 \cdot \left(1 + \frac{(x-9)}{1!} + \frac{(x-9)^2}{2!} + \frac{(x-9)^3}{3!} + O((x-9)^3)\right).$$

Remark

The finite expansion of function $f(x)$ at infinity is given by

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n} + O\left(\frac{1}{x^n}\right).$$

4.7 Applications of finite expansions

Finite expansions are valuable tools for understanding the behaviour of a function near a specified point, and they prove to be particularly useful when dealing with limits, especially when faced with indeterminate forms. When taking the limit as x approaches a particular point, such as $x \to 0$, finite expansions allow us to simplify expressions by replacing them with finite expansions.

Example 4.3

Find the limit of

$$\lim_{x \to 0} \frac{\sin(9x)}{\sinh(-4x)}$$

We have

$$\lim_{x\to 0}\frac{\sin(9x)}{\sinh(-4x)} = \frac{0}{0}, \text{ indeterminate forms}$$

$$\sin(x) = x + o(x^2) \implies \sin(9x) = 9x + o(x^2)$$

$$\sinh(x) = x + o(x^2) \implies \sinh(-4x) = -4x + o(x^2)$$

then

$$\lim_{x\to 0}\frac{\sin(9x)}{\sinh(-4x)} = \lim_{x\to 0}\frac{9x + o(x^2)}{-4x + o(x^2)} = -\frac{9}{4}$$

Example 4.4

Find the first three terms of the finite expansion for $\sin x$ and $\cos x$. Hence find

$$\lim_{x\to 0}\frac{1-\cos(\sin x)}{x^2}$$

The finite expansion

- $\sin x = x - \frac{x^3}{3!} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2!} + o(x^3)$

We have

$$\cos(\sin x) = 1 - \frac{(x - \frac{x^3}{3!})^2}{2!} + o(x^3)$$
$$= 1 - \frac{x^2}{2} + o(x^3)$$

Thus

$$\lim_{x\to 0}\frac{1-\cos(\sin x)}{x^2} = \lim_{x\to 0}\frac{1-(1-\frac{x^2}{2}+o(x^3))}{x^2}$$
$$= \lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^3)}{x^2}$$
$$= \frac{1}{2}$$

Chapter 4 Exercises

Exercise 1

- Find the Maclaurin series expansion of the following functions:

$$e^x,\ \cos x,\ \ln(x+1)$$

- Find the finite series expansion of the following functions at the vicinity of zero of order 2, then conclude the value of $f'(0)$ and $f''(0)$

$$f(x) = e^{2x} + \frac{1}{x-1} - 3x$$

- Using the finite series expansion find the following limit:

$$\lim_{x\to 0}\frac{x - x\cos x}{x - \sin x}$$

Exercise 2

Find the finite expansion of

$$\frac{x^3\sin(2x)-2x}{x^3},\ \frac{e^x-e^{-x}}{x},\ \sinh(x)$$

Exercise 3

Let $f(x)$ be the function defined by:

$$f(x)=\frac{e^{\cos(x)}-1}{x\tan(x)},\quad \forall x\in\left]-\frac{\pi}{2},0\right[\cup\left]0,\frac{\pi}{2}\right[$$

- Find the third-order Taylor expansion of f at zero.
- Compute the limit:

$$\lim_{x\to 0}\frac{e^{\cos(x)}-1}{x\tan(x)}$$

Exercise 4

- Find a finite expansion of order 2 near 0 for the functions:

$$h(x)=e^{x-x^2},\ f(x)=\frac{\ln(1+x)}{\sqrt{1+x}}$$

- Using the above expansions, compute the limit:

$$\lim_{x\to 0}\left(\frac{h(x)-f(x)-1}{x^2}\right)$$

- Let the function g defined on $\mathbb{R}^*$ by:

$$g(x)=x^2\sqrt{\frac{x}{1+x}}\ln\left(\frac{1+x}{x}\right)$$

- Deduce a finite expansion near $[0,+\infty)$ for the function g.

Note: Observe that

$$\frac{1}{t}=\frac{1}{t^2}\cdot t$$

Exercise 5

Let

$$f(x)=\frac{e^{2\cosh x}-x\ln(\cos x)-(1+2x)^{\frac{1}{x}}}{1-\sqrt{1-2x}}$$

- Give the finite expansion of f up to order 2 in a neighborhood of 0.
- Deduce that f is extendable by continuity at 0. Let g be this extension.
- Show that g is differentiable at 0 and calculate $g'(0)$.
- Give the equation of the tangent to the curve of g at $x=0$ and determine their relative position near 0.

Exercise 6

Let $f(x) = \left(2e^x - \cosh(x\sqrt{2})\right)^{\frac{1}{\sinh x}}$.

- Give the finite expansion of f up to order 2 in a neighborhood of 0.
- Show that f is extendable by continuity at 0. Let g be this extension.
- Give the equation of the tangent (T) to the curve (C) of g at $x = 0$ and find their relative position in a neighborhood of 0.

Exercise 7

Let h be the function defined on $\mathbb{R}$ by $h(x) = \arcsin\left(\frac{1-x^2}{1+x^2}\right)$.

- Calculate $h'(x)$ for all $x \in \mathbb{R}^*$. Deduce that

$$h(x) = \begin{cases} -2\arctan x + \frac{\pi}{2} & \text{if } x > 0 \\ 2\arctan x + \frac{\pi}{2} & \text{if } x < 0 \end{cases}$$

- Deduce the finite expansion of $h(x)$ to order 3 as $x \to 0^+$.

Let $f(x) = \frac{e^{\tan x} - \sinh\left(\frac{\sqrt[3]{1+3x^2}-1}{x}\right) - \cosh x}{h(x)+2x-\frac{\pi}{2}}$. Calculate $\lim_{x\to 0^+} f(x)$.

Chapter 5 Integrals

In this chapter, our objective is to empower students with foundational concepts in integral calculus, providing a comprehensive array of integration techniques that will be useful throughout the remainder of this semester's program. Integration is a central tool in mathematics, allowing us to recover functions from their derivatives and to solve problems involving areas, volumes, and other applications. We begin with antiderivatives of elementary functions and then introduce techniques of integration, including substitution, trigonometric methods, and integration by parts. By the end of this chapter, students will be able to:

- Understand the meaning of an indefinite integral.
- Compute antiderivatives of elementary functions.
- Use the method of integration by parts.
- Evaluate trigonometric integrals.
- Apply different techniques of integration to solve problems.

5.1 Indefinite integrals

Definition 5.1

Let f be a function defined on a closed interval $[a, b]$ on $\mathbb{R}$, and let F be a differentiable function on $[a, b]$. F is called a primitive or antiderivative of f on $[a, b]$ if for all x on $[a, b]$, $F'(x) = f(x)$.

Proposition 5.1

Let $F_1(x)$ and $F_2(x)$ be primitives of f on $[a, b]$, meaning they are antiderivatives of f on $[a, b]$. Then, for all x in $[a, b]$, there exists a constant C such that $F_1(x) = F_2(x) + C$.

Definition 5.2

The set of all primitives of the function $f : [a, b] \to \mathbb{R}$ is called the indefinite integral of f, denoted by $\int f(x)\,dx$, so if F is a primitive of f on $[a, b]$, we have

$$\int f(x)\,dx = F(x) + c, \quad c \in \mathbb{R}.$$

In this definition, the $\int$ is called the integral symbol, $f(x)$ is called the integrand, x is called the integration variable, and the "c" is called the constant of integration.

Example 5.1

- $\int e^x = e^x + c,\ c \in \mathbb{R}$.
- $\int \frac{1}{x} = \ln x + c,\ c \in \mathbb{R}$.

Theorem 5.1

Let f be a continuous function on $[a,b]$. For any primitive F of f, we have:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Proposition 5.2

Let f and g be two continuous functions on $[a,b]$. We have:

- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.
- $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.
- $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$; *for* $\alpha \in \mathbb{R}$.
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$; *for* $c \in [a,b]$.
- $\int_a^a f(x)\,dx = 0$.

Remark

Note that $\int_a^b [f(x) \cdot g(x)]\,dx \neq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$.

5.1.1 Antiderivatives of elementary functions

Function	Primitive function	Interval
x^α, $\alpha \neq -1$	$\frac{x^{\alpha+1}}{\alpha+1}$	$\mathbb{R} - \{0\}$
e^x	e^x	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\frac{1}{1+x^2}$	$\arctan x$	$\mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$]-1,1[$

5.1.2 Basic integration formulas

Integral	Formula
$\int du$	$u + C$
$\int \alpha\, du$	$\alpha u + C$
$\int \tan u\, du$	$-\ln\lvert\cos u\rvert + C$
$\int \csc u \cot u\, du$	$-\csc u + C$
$\int \sec u \tan u\, du$	$\sec u + C$
$\int \csc^2 u\, du$	$-\cot u + C$
$\int \sec^2 u\, du$	$\tan u + C$
$\int \cos u\, du$	$\sin u + C$
$\int \sin u\, du$	$-\cos u + C$
$\int \frac{du}{u}$	$\ln\lvert u\rvert + C$
$\int u^n\, du$	$\frac{u^{n+1}}{n+1} + C$ for $n \neq -1$
$\int \frac{du}{u^2-a^2}$	$\cosh^{-1}\frac{u}{a} + C$ for $u > a > 0$
$\int \frac{du}{a^2+u^2}$	$\sinh^{-1}\frac{u}{a} + C$ for $a > 0$
$\int \frac{du}{u\sqrt{u^2-a^2}}$	$\frac{1}{a}\sec^{-1}\left(\frac{a}{u}\right) + C$
$\int \frac{du}{a^2+u^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{du}{a^2-u^2}$	$\sin^{-1}\left(\frac{u}{a}\right) + C$
$\int \cosh u\, du$	$\sinh u + C$
$\int \sinh u\, du$	$\cosh u + C$
$\int e^u\, du$	$e^u + C$
$\int \cot u\, du$	$\ln\lvert\sin u\rvert + C$

5.2 Techniques of integration

5.2.1 Integration by parts

Theorem 5.2

Let u and v be two differentiable functions of class C^1 on $[a,b]$. We have

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

Remark

In some examples, it is necessary to apply this method several times to obtain the result.

Example 5.2

1. $I_1 = \int xe^x \, dx$
   We use integration by parts, we can choose $u = x$ and $dv = e^x$. Then, we find du and v:
   $$du = dx, \quad v = e^x$$
   Now, we apply the integration by parts formula:
   $$I_1 = xe^x - \int e^x \, dx$$
   So,
   $$I_1 = xe^x - e^x + C, \, C \in \mathbb{R}$$
   Simplifying, we get:
   $$I_1 = e^x(1-x) + C, \, C \in \mathbb{R}$$
2. $I_2 = \int x^2 e^{9x} \, dx$
   Choose $u = x^2$ and $dv = e^{9x} \, dx$. Thus, $du = 2x \, dx$ and $v = \int e^{9x} \, dx = \frac{1}{9}e^{9x}$. Therefore,
   $$u = x^2, \quad dv = e^{9x} \, dx, \quad du = 2x \, dx, \quad v = \frac{1}{9}e^{9x}.$$
   We apply the integration by parts formula:
   $$I_2 = \frac{1}{9}x^2 e^{9x} - \int \frac{2}{9}xe^{9x} \, dx.$$
   We still cannot integrate $\int \frac{2}{9}xe^{9x} \, dx$ directly, but the integral now has a lower power on x. We can evaluate this new integral by using integration by parts again. To do this, choose $u = x$ and $dv = \frac{2}{9}e^{9x} \, dx$. Thus, $du = dx$ and $v = \int \frac{2}{9}e^{9x} \, dx = \frac{2}{81}e^{9x}$. Now we have
   $$u = x, \quad dv = \frac{2}{9}e^{9x} \, dx, \quad du = dx, \quad v = \frac{2}{81}e^{9x}.$$
   Substituting back into the previous equation yields
   $$I_2 = \frac{1}{9}x^2 e^{9x} - \left(\frac{2}{81}xe^{9x} - \int \frac{2}{81}e^{9x} \, dx \right).$$
   After evaluating the last integral and simplifying it, we obtain
   $$I_2 = \frac{1}{9}x^2 e^{9x} - \frac{2}{81}xe^{9x} + \frac{2}{729}e^{3x} + C.$$

5.2.2 Integration by substitution

5.2.2.1 Integration by substituting $u = ax + b$

We introduce the technique through some simple examples where a linear substitution is suitable.

Example 5.3

1. $I_1 = \int (x+9)^4 \, dx$
   In the integral I_1, the power of 4 makes it more complex, compounded by the term $x + 9$. To address this, we employ a substitution.
   Let $u = x + 9$. This change simplifies the integral to u^4. However, we must appropriately

account for the differential dx.
Expressing differentials, we have

$$du = \left(\frac{du}{dx}\right) dx$$

For this example, since $u = x + 9$, we immediately have $\frac{du}{dx} = 1$, yielding $du = dx$.
Substituting both u and du in I_1, we get

$$\int (x+9)^4 \, dx = \int u^4 \, du$$

The resulting integral is $\frac{u^5}{5} + C$. Reverting to x by recalling $u = x + 9$, we have

$$I_1 = \frac{(x+9)^5}{5} + C$$

Integration by substitution is now complete.

2. $I_2 = \int \cos(x+4)\, dx$
If we set $u = x + 4$, then

$$du = dx$$

Substituting both u and du in I_2, we have

$$I_2 = \int \cos u \, du = \sin u + C$$

So, we can revert to an expression involving the original variable x by recalling that $u = x + 4$, giving

$$I_2 = \sin(x+4) + C$$

5.2.2.2 $\int f(g(x)) \cdot g'(x)\, dx$ by substituting $u = g(x)$

Given that F and g are differentiable functions, the chain rule for differentiation states:

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x)) \cdot g'(x).$$

If $F'(x) = f(x)$, meaning F is an antiderivative of f, then this simplifies to:

$$\frac{d}{dx}\left(F(g(x))\right) = f(g(x)) \cdot g'(x).$$

In other words, if F is an antiderivative of f, then:

$$\int f(g(x)) \cdot g'(x)\, dx = F(g(x)) + C.$$

Now, let's simplify this further. Let $g(x) = u$, so $g'(x) = \frac{du}{dx}$. Multiplying both sides by dx, we get:

$$g'(x)\, dx = du.$$

We substitute $g(x)$ with u and $g'(x)\, dx$ with du:

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du = F(u) + C.$$

As a result, the Substitution Rule is given by

$$\text{If } u = g(x), \text{then } \int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

Example 5.4

1. $I_1 = \int x^2(x^3+9)^4\, dx$
   Let $u = x^3 + 9$, so that $du = 3x^2\, dx$.
$$\begin{aligned} I_1 &= \int x^2(x^3+9)^4\, dx \\ &= \int (x^3+9)^4 \cdot (3 \cdot x^2) \cdot \frac{1}{3}\, dx \end{aligned} \tag{5.1}$$
   So, substituting u for $u = x^3 + 9$, and with $du = 3x^2\, dx$ in Equation (5.1) we have
$$\begin{aligned} I_1 &= \frac{1}{3}\int u^4\, du \\ &= \frac{1}{3}(\frac{u^5}{5}) + C \\ &= \frac{(x^3+9)^5}{15}) + C \end{aligned}$$
2. $I_2 = \int 2x\sqrt{1+x^2}\, dx$
   Let $u = 1 + x^2$. From this, we get $du = 2x$ So,
$$\begin{aligned} I_2 &= \int 2x\sqrt{1+x^2}\, dx \\ &= \int \sqrt{u}\, du \\ &= \int u^{\frac{1}{2}}\, du \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C \end{aligned}$$

5.2.2.3 Integration by partial fractions

Consider a rational function $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials and the degree of $h(x)$ is greater than the degree of $g(x)$. To integrate such a rational function using partial fractions, we first need to decompose it into simpler fractions. Before setting up the decomposition, it's essential to factorize the denominator. Here we present the partial fractions method of Partial Fractions:

1. Let $(x - r)$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $(x - r)$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be an irreducible quadratic factor of $g(x)$ so that x^2+px+q has no real roots. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\frac{B_3x+C_3}{(x^2+px+q)^3}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$.
3. Set the original fraction $\frac{g(x)}{h(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equating the coefficients of corresponding powers of x, solve the resulting equations for the undetermined coefficients.

Remark

- The denominator is a product of linear factors, with none repeating. In this case, the partial fraction decomposition takes the form:
$$\frac{x+1}{(x-4)(4x-9)}=\frac{A}{x-4}+\frac{B}{4x-9}$$
- The denominator consists of linear factors, with some repeating. The partial fraction decomposition looks like this:
$$\frac{x+1}{(x-4)(x-9)^3}=\frac{A}{x-4}+\frac{B}{x-9}+\frac{C}{(x-9)^2}+\frac{D}{(x-9)^3}$$
- The denominator contains irreducible quadratic factors, with none repeating. The partial fraction decomposition becomes:
$$\frac{x+1}{(x-4)^2(x^2+9)}=\frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{Cx+D}{x^2+9}$$
- The denominator includes irreducible quadratic factors, with some repeating. The corresponding partial fraction decomposition is:
$$\frac{x+1}{(x-4)(x^2+9)^2}=\frac{A}{x-4}+\frac{Bx+C}{x^2+9}+\frac{Dx+E}{(x^2+9)^2}$$

5.2.2.4 Strategy for evaluating $\int \frac{1}{(x-a)^n}\,dx$

- For $n=1$ we have $\int \frac{1}{x-a}\,dx=\ln|x-a|+C$
- For $n>1$ we get $\int \frac{1}{(x-a)^n}\,dx=\frac{-1}{(n-1)(x-a)^{n-1}}+C$

5.2.2.5 Strategy for evaluating $\int \frac{1}{ax^2+bx+c}\,dx$

To evaluate the integral $\int \frac{1}{ax^2+bx+c}\,dx$, first calculate $\Delta=b^2-4ac$.

1. If $\Delta<0$:
Rewrite the expression as
$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2+\beta^2\right]$$

where $\beta^2 = \frac{-\Delta}{4a^2}$. Subsequently,

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2+\beta^2}\,dx$$

Proceed to solve this integral using the substitution $u = x + \frac{b}{2a}$. Therefore,

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a\beta}\arctan\left(\frac{x+\frac{b}{2a}}{\beta}\right)+C$$

2. If $\Delta = 0$:
   Rewrite the expression as

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2$$

   Then

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2}\,dx$$

   Let $u = x+\frac{b}{2a}$, then $du = dx$. Thus,

$$\begin{aligned}\int \frac{1}{ax^2+bx+c}\,dx &= \frac{1}{a}\int \frac{1}{u^2}\,du\\ &= \frac{1}{a}\int u^{-2}\,du\\ &= \frac{1}{a}\frac{u^{-1}}{-1}+C\\ &= \frac{-1}{au}+C\\ &= \frac{-1}{a\left(x+\frac{b}{2a}\right)}+C\\ &= \frac{-1}{\left(ax+\frac{b}{2}\right)}+C\\ &= \frac{-2}{2ax+b}+C\end{aligned}$$

3. If $\Delta > 0$:
   Rewrite the expression as

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

   Then

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{(x-x_1)(x-x_2)}\,dx$$

   Then

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\left(\int \frac{A}{(x-x_1)}\,dx + \int \frac{B}{(x-x_2)}\,dx\right)$$

So,

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\left(\int \frac{\frac{1}{(x_1-x_2)}}{(x-x_1)}\,dx + \int \frac{\frac{1}{(x_1-x_2)}}{(x-x_2)}\,dx\right)$$
$$= \frac{1}{a(x_1-x_2)}\ln\left|\frac{x-x_1}{x-x_2}\right| + C$$

Example 5.5

1. $I_1 = \int \frac{x+14}{(x+5)(x+2)}\,dx$
   Our first step is to decompose $\frac{x+14}{(x+5)(x+2)}$ as:
   $$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}.$$
   We want to find constants A and B for all $x \neq -5$ and $x \neq -2$.
   $$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2}.$$
   We solve for A and B by cross-multiplying and equating the numerators:
   $$\frac{x+14}{(x+5)(x+2)} = \frac{A}{x+5} + \frac{B}{x+2} = \frac{A(x+2)+B(x+5)}{(x+5)(x+2)}$$
   Then
   $$x+14 = A(x+2)+B(x+5)$$
   $$= Ax+2A+Bx+5B$$
   $$= (A+B)x+2A+5B$$
   So, we get
   $$A+B = 1\ldots(1)$$
   $$2A+5B = 14\ldots(2)$$
   From (1) we obtain $B = 1-A$, Substituting this into (2) we get
   $$14 = 2A+5B$$
   $$= 2A+5(1-A)$$
   $$= 2A+5-5A$$
   $$= 5-3A$$
   Then
   $$A = -3 \text{ and } B = 4$$
   So,
   $$\int \frac{x+14}{(x+5)(x+2)}\,dx = \int\left(\frac{-3}{x+5}+\frac{4}{x+2}\right)dx = -3\ln|x+5| + 4\ln|x+2| + C$$
2. $I_2 = \int \frac{6x+7}{(x+2)^2}\,dx$
   Our first step is to decompose $\frac{6x+7}{(x+2)^2}$ as:
   $$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

Then, multiply both sides by $(x+2)^2$.

$$\begin{aligned} 6x+7 &= A(x+2)+B \\ &= Ax+2A+B \end{aligned}$$

Equating coefficients of corresponding powers of x gives $A = 6$ and $B = -5$.

Therefore,

$$\begin{aligned} I_2 &= \int (\frac{6}{x+2} - \frac{5}{(x+2)^2})\,dx \\ &= 6\int \frac{1}{x+2}\,dx - 5\int \frac{1}{(x+2)^2}\,dx \\ &= 6\ln|x+2| + \frac{5}{x+2} + C \end{aligned}$$

3. $I_3 = \int \frac{1}{x^2-2x+5}\,dx$

We have $\Delta = -16$, let $x^2-2x+5 = (x-1)^2+4$

Now, substitute $u = x-1$, we have $du = dx$. Thus,

$$\begin{aligned} I_3 &= \int \frac{1}{(x-1)^2+4}\,dx \\ &= \int \frac{1}{u^2+4}\,du \\ &= \frac{1}{2}\arctan\frac{u}{2} + C \\ &= \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C \end{aligned}$$

5.2.2.6 Trigonometric integrals

Strategy for evaluating $\int \sin^m(x)\cos^n(x)\,dx$

- If the power n of cosine is odd ($n = 2k+1$), save one cosine factor and use $\cos^2(x) = 1-\sin^2(x)$ to express the rest of the factors in terms of sine:

$$\begin{aligned} \int \sin^m(x)\cos^n(x)\,dx = \int \sin^m(x)\cos^{2k+1}(x)\,dx &= \int \sin^m(x)(\cos^2(x))^k\cos(x)\,dx \\ &= \int \sin^m(x)(1-\sin^2(x))^k\cos(x)\,dx \end{aligned}$$

  Then solve by substitution and let $u = \sin(x)$.

- If the power m of sine is odd ($m = 2k+1$), save one sine factor and use $\sin^2(x) = 1-\cos^2(x)$ to express the rest of the factors in terms of cosine:

$$\begin{aligned} \int \sin^m(x)\cos^n(x)\,dx = \int \sin^{2k+1}(x)\cos^n(x)\,dx &= \int (\sin^2(x))^k\cos^n(x)\sin(x)\,dx \\ &= \int (1-\cos^2(x))^k\cos^n(x)\sin(x)\,dx \end{aligned}$$

  Then solve by substitution and let $u = \cos(x)$.

- If both powers m and n are even, use the half-angle identities:

$$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}, \quad \cos^2\left(\frac{x}{2}\right) = \frac{1+\cos(x)}{2}$$

Example 5.6

1. $I_1 = \int \cos^3 x\, dx$

   Here we can separate one cosine factor and convert the remaining factor to an expression involving sine using $\cos^2 x = 1 - \sin^2 x$.

$$\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x$$

   Let $u = \sin x$, since then $du = \cos x\, dx$. Thus,

$$\begin{aligned} I_1 &= \int (1 - \sin^2 x) \cos x\, dx \\ &= \int (1 - u^2)\, du \\ &= u - \frac{u^3}{3} + C \\ &= \sin x - \frac{\sin^3 x}{3} + C \end{aligned}$$

2. $I_2 = \int \cos^8 x \sin^5 x\, dx$

   Since the power on $\sin x$ is odd, we have

$$\cos^8 x \sin^5 = \cos^8 x \sin^4 x \sin x$$

   Rewrite $\sin^4 x = (\sin^2 x)^2$

$$\cos^8 x \sin^5 = \cos^8 x (\sin^2 x)^2 \sin x$$

   Now, substitute $\sin^2 x = 1 - \cos^2 x$, we obtain

$$\cos^8 x \sin^5 = \cos^8 x (1 - \cos^2 x)^2 \sin x$$

   Then

$$I_2 = \int \cos^8 x (1 - \cos^2 x)^2 \sin x\, dx$$

   Let $u = \cos x$ and $du = -\sin x\, dx$.Thus

$$\begin{aligned} I_2 &= \int \cos^8 x (1 - \cos^2 x)^2 \sin x\, dx \\ &= \int u^8 (1 - u^2)^2 (-du) \\ &= \int \left(-u^8 + 2u^{10} - u^{12}\right)\, du \\ &= -\frac{1}{9} u^9 + \frac{2}{11} u^{11} - \frac{1}{13} u^{13} + C \\ &= -\frac{1}{9} \cos^9 x + \frac{2}{11} \cos^{11} x - \frac{1}{13} \cos^{13} x + C. \end{aligned}$$

3. $I_3 = \int \sin^4 x\, dx$.

   We have

$$\sin^4 x = (\sin^2 x)^2$$

Since $\sin^2 x$ occurs, we could evaluate integral I_3 using the formula:

$$\sin^2 x = \frac{1-\cos 2x}{2}.$$

This gives:

$$\begin{aligned} I_3 &= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx \end{aligned}$$

Since $\cos^2 2x$ occurs, we must use another formula

$$\cos^2\, 2x = \frac{1+\cos\, 4x}{2}.$$

Then

$$\begin{aligned} I_3 &= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx \\ &= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx \\ &= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx \\ &= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C \end{aligned}$$

1. Strategy for evaluating $\tan^m(x)\sec^n(x)\,dx$
   - If the power n of secant is even ($n = 2k$, $k \geq 2$), save one $\sec^2(x)$ factor and use $\sec^2(x) = 1 + \tan^2(x)$ to express the rest of the factors in terms of tangent:
   $$\begin{aligned} \int \tan^m(x)\sec^n(x)\,dx = \int \tan^m(x)\sec^{2k}(x)\,dx &= \int \tan^m(x)(\sec^2)^{k-1}\sec^2(x)\,dx \\ &= \int \tan^m(x)(1+\tan^2(x))^{k-1}\sec^2(x)\,dx \end{aligned}$$
   Then solve by substitution and let $u = \tan(x)$.
   - If the power m of tangent is odd ($m = 2k+1$), save one $\sec(x)\tan(x)$ factor and use $\tan^2(x) = \sec^2(x) - 1$ to express the rest of the factors in terms of secant:
   $$\begin{aligned} \int \tan^m(x)\sec^n(x)\,dx = \int \tan^{2k+1}(x)\sec^n(x)\,dx &= \int (\tan^2(x))^k \sec^{n-1}(x)\sec(x)\tan(x)\,dx \\ &= \int (\sec^2(x)-1)^k \sec^{n-1}(x)\sec(x)\tan(x)\,dx \end{aligned}$$
   Then solve by substitution and let $u = \sec(x)$.

Example 5.7

1. $I_1 = \int \tan^6(x)\sec^4(x)\,dx$

   Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,
   $$I_1 = \int \tan^6 x \left(\tan^2 x + 1\right)\sec^2 x\,dx$$

Let $u = \tan x$ and $du = sec^2 x$. Then

$$\begin{aligned} I_1 &= \int \tan^6 x \left(\tan^2 x + 1\right) \sec^2 x \, dx \\ &= \int u^6 \left(u^2 + 1\right) \, du \\ &= \int (u^8 + u^6) \, du \\ &= \frac{1}{9} u^9 + \frac{1}{7} u^7 + C \quad (\text{Substitute } \tan x = u) \\ &= \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C. \end{aligned}$$

2. $I_2 = \int \tan^5 x \sec^3 x \, dx$

Since the power on $\tan x$ is odd, we can write

$$\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$$

Now we write $\tan^4 x = (\tan^2 x)^2$, we obtain

$$\tan^5 x \sec^3 x = (\tan^2 x)^2 \sec^2 x \sec x \tan x$$

Then,

$$I_2 = \int (\tan^2 x)^2 \sec^2 x \sec x \tan x \, dx$$

Using $\tan^2 x = \sec^2 x - 1$, we get

$$\begin{aligned} I_2 &= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x \, dx \end{aligned}$$

Let $u = \sec x$ and $du = \sec x \tan x \, dx$

$$\begin{aligned} I_2 &= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1)^2 \sec^2 x \, du \\ &= \int \left(u^6 - 2u^4 + u^2\right) \, du \\ &= \frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 + C. \\ &= \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C. \end{aligned}$$

Chapter 5 Exercises

Exercise 1

Calculate the following integrals:

$$\int_0^2 (9x+1)^3\,dx, \quad \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{1+x^2}\,dx, \quad \int_0^2 \frac{1}{4+x^2}\,dx,$$

$$\int_1^2 \frac{1}{1+4x}\,dx, \quad \int \frac{dx}{x^2-9}, \quad \int \frac{dx}{\sqrt{4+x^2}}\,dx.$$

Exercise 2

- Using variable substitution, compute the following integrals:

$$\int \sin^2 x\cos x\,dx, \quad \int e^{\sin x}\cos x\,dx, \int x^2(x^3-9)^5\,dx, \quad \int \frac{\ln(9+x)}{9+x}\,dx.$$

- Using integration by parts, calculate the following integrals:

$$\int xe^{-x}\,dx, \quad \int (x^2+4x+9)e^{-x}\,dx, \quad \int \arctan x\,dx,$$

$$\int e^x \sin x\,dx, \quad \int e^{-x}\sin 2x\,dx, \quad \int x^2\ln x\,dx.$$

Exercise 3

- Compute the integral:

$$I = \int \frac{4x^2+3x-2}{(x+1)(x^2+1)}\,dx$$

- Using integration by substitution, compute the integral:

$$J = \int_0^{\pi/2} \sin(x)\cos^2(x)\,dx$$

- Compute the integral using trigonometric substitution:

$$L = \int \frac{1}{\sqrt{9-x^2}}\,dx$$

Exercise 4

Evaluate each of the following integrals:

- $\int \frac{x^2}{(x-4)(x-9)}\,dx.$
- $\int \frac{x-4}{x(x-2)^3}\,dx.$
- $\int \frac{x^2+4x+9}{x^2+1}\,dx.$

Chapter 6 Solutions to selected exercises

6.1 Chapter 1 selected solutions

Exercise 1

Prove the following properties:

- $\forall x, y \in \mathbb{R},\ |x+y| \leq |x| + |y|$.
- $\forall x, y \in \mathbb{R},\ \big||x| - |y|\big| \leq |x-y|$.
- $\forall x, y \in \mathbb{R},\ |x| + |y| \leq |x+y| + |x-y|$.
- $\forall x \in \mathbb{R},\ |x| = \max\{x, -x\}$.

Solution

1. $\forall x, y \in \mathbb{R} : |x+y| \leq |x| + |y|$

   We begin by considering:

$$\forall x \in \mathbb{R}, \quad -|x| \leq x \leq |x| \quad \ldots(1)$$
$$\forall y \in \mathbb{R}, \quad -|y| \leq y \leq |y| \quad \ldots(2)$$

   Adding inequalities (1) and (2):

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

   By the definition of absolute value:

$$|x+y| \leq |x| + |y|$$

   Thus, the inequality holds.

2. $\forall x, y \in \mathbb{R} : ||x| - |y|| \leq |x-y|$

   Using the triangle inequality:

$$|x| = |x+y-y| \leq |x-y| + |y| \quad \ldots(1)$$

   Similarly:

$$|y| = |y+x-x| \leq |y-x| + |x| \quad \ldots(2)$$

   Combining (1) and (2):

$$|x| - |y| \leq |x-y| \quad \textit{and} \quad |y| - |x| \leq |x-y|$$

Therefore:

$$-|x-y| \leq |x| - |y| \leq |x-y|$$

Hence, by the definition of absolute value:

$$||x| - |y|| \leq |x-y|$$

3. $\forall x, y \in \mathbb{R} : |x| + |y| \leq |x+y| + |x-y|$
We start with:

$$2x = (x+y) + (x-y), \quad 2y = (x+y) - (x-y)$$

By the triangle inequality:

$$2|x| \leq |x+y| + |x-y| \quad \ldots(1)$$
$$2|y| \leq |x+y| + |x-y| \quad \ldots(2)$$

Adding (1) and (2) gives:

$$|x| + |y| \leq |x+y| + |x-y|$$

4. $\forall x \in \mathbb{R} : |x| = \max\{x, -x\}$
- If $x \geq 0$: By the definition of absolute value, $|x| = x$, and $-x \leq x$. Therefore, $x = \max(x, -x)$. - If $x \leq 0$: Then $|x| = -x$, and $-x \geq x$. Hence, $-x = \max(x, -x)$.
Thus, we conclude that:

$$|x| = \max(x, -x)$$

Exercise 2

If the set A is bounded, find $\sup A$, $\max A$, $\inf A$, and $\min A$ if they exist.

$$A = \{x \in \mathbb{R} : 0 < x < 9\}, \quad A = \left\{9 - \frac{1}{n}, n \in \mathbb{N}^*\right\},$$
$$A = \{x \in \mathbb{R} : x^3 > 64\}, \quad A = \left\{\frac{1}{x} : 4 \leq x \leq 9\right\},$$
$$A = \left\{\frac{n+2}{n-1}, n \in \mathbb{N}, n \geq 2\right\}, \quad A = \left\{9 + \frac{1}{n}, n \in \mathbb{N}^*\right\}$$

Solution

1. $A = \{x \in \mathbb{R} \mid 0 < x < 9\} =]0, 9[$
   - *Supremum ($\sup A$): The supremum of a set is its least upper bound. The set of upper bounds is $]9, +\infty[$. The smallest upper bound is 9, so $\sup A = 9$.*
   - *Maximum ($\max A$): The maximum is the largest element in the set. Since $5 \notin A$, A has no maximum.*
   - *Infimum ($\inf A$): The infimum is the greatest lower bound. The set of lower bounds is $]-\infty, 0[$. The greatest lower bound is 0, so $\inf A = 0$.*
   - *Minimum ($\min A$): Since $0 \notin A$, the set has no minimum.*
2. $A = \left\{9 - \frac{1}{n} \mid n \in \mathbb{N}^*\right\}$

Let $a_n = 9 - \frac{1}{n}$*. Then* A *is nonempty and bounded:*

$$\forall n \in \mathbb{N}^*, \quad 8 < a_n < 9.$$

By the completeness axiom, $\sup A$ *and* $\inf A$ *exist.*

- *Supremum* $(\sup A)$*:* 9 *is an upper bound of* A*. We show it is the least upper bound using the characterization property:*

  $$\sup A = 9 \iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^* \text{ such that } 9 - \varepsilon < a_{n_\varepsilon}.$$

  Since $9 - \varepsilon < 9 - \frac{1}{n_\varepsilon}$ *implies* $n_\varepsilon > \frac{1}{\varepsilon}$*, by the Archimedean property such* n_ε *exists. Hence,* $\sup A = 9$.
- *Maximum* $(\max A)$*: Since* $9 \notin A$*,* A *has no maximum.*
- *Infimum* $(\inf A)$*: For all* $n \in \mathbb{N}^*$*, we have* $9 - \frac{1}{n} \geq 8$*, so* $\inf A = 8$.
- *Minimum* $(\min A)$*: Since* $8 \in A$ *(when* $n = 1$*),* $\min A = 8$.

3. $A = \{x \in \mathbb{R} \mid x^3 > 64\} =]4, +\infty[$
   - *Supremum* $(\sup A)$*:* A *is unbounded above, so* $\sup A$ *does not exist.*
   - *Maximum* $(\max A)$*: Since* $\sup A$ *does not exist,* $\max A$ *does not exist either.*
   - *Infimum* $(\inf A)$*: The set of lower bounds is* $]-\infty, 4[$*, so* $\inf A = 4$.
   - *Minimum* $(\min A)$*: Since* $4 \notin A$*,* A *has no minimum.*
4. $A = \left\{\frac{1}{x} \mid x \in [1, 2]\right\}$

   The function $f(x) = \frac{1}{x}$ *is decreasing on* $[4, 9]$*. So:*

   $$\frac{1}{9} \leq f(x) \leq \frac{1}{4}$$

   - *Supremum* $(\sup A)$*: The smallest upper bound is* $\frac{1}{4}$*, so* $\sup A = \frac{1}{4}$.
   - *Maximum* $(\max A)$*: Since* $\frac{1}{4} \in A$*,* $\max A = \frac{1}{4}$.
   - *Infimum* $(\inf A)$*: The greatest lower bound is* $\frac{1}{9}$.
   - *Minimum* $(\min A)$*: Since* $\frac{1}{9} \in A$*,* $\min A = \frac{1}{9}$.
5. $A = \left\{\frac{n+2}{n-1} \mid n \in \mathbb{N}, n \geq 2\right\}$

   The first few values of A *are:* $4, \frac{5}{2}, \ldots$

   $$\forall n \geq 2, \quad 1 < a_n \leq 4$$

   - *Supremum* $(\sup A)$*: The smallest upper bound is* 4*, so* $\sup A = 4$.
   - *Maximum* $(\max A)$*: Since* $4 \in A$*,* $\max A = 4$.
   - *Infimum* $(\inf A)$*: To show* $\inf A = 1$*, use:*

     $$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ such that } a_{n_\varepsilon} < 1 + \varepsilon.$$

     Solving $1 + \varepsilon > 1 + \frac{3}{n_\varepsilon - 1}$ *gives* $n_\varepsilon > \frac{3}{\varepsilon} + 1$*, which exists by the Archimedean property. Hence,* $\inf A = 1$.
   - *Minimum* $(\min A)$*: Since* $1 \notin A$*,* A *has no minimum.*
6. $A = \left\{9 + \frac{1}{n} \mid n \in \mathbb{N}^*\right\}$

   $$\forall n \in \mathbb{N}^*, \quad 9 < a_n \leq 10$$

- *Supremum* ($\sup A$)*: The smallest upper bound is* 10*, so* $\sup A = 10$.
- *Maximum* ($\max A$)*: Since* $10 \in A$ *(when* $n = 1$*),* $\max A = 10$.
- *Infimum* ($\inf A$)*: To show* $\inf A = 9$*, use:*

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}^* \text{ such that } a_{n_\varepsilon} < 9 + \varepsilon.$$

Solving $9 + \varepsilon > 9 + \frac{1}{n_\varepsilon}$ *gives* $n_\varepsilon > \frac{1}{\varepsilon}$*, which exists. Hence,* $\inf A = 9$.
- *Minimum* ($\min A$)*: Since* $9 \notin A$*, the set has no minimum.*

Exercise 3

Find the sup, max, inf and min of the following sets and prove your answer.

- $A = \left\{\frac{8}{n^2+4};\ n \in \mathbb{N}\right\}$
- $A = \left\{\frac{2n+1}{n+1};\ n \in \mathbb{N}\right\}$

Solution

- $A = \left\{\frac{8}{n^2+4};\ n \in \mathbb{N}\right\}$

We have

$$\forall n \in \mathbb{N} : n^2 \geq 0 \Rightarrow n^2 + 4 \geq 4 \Rightarrow 0 < \frac{1}{n^2+4} \leq \frac{1}{4} \Rightarrow 0 < \frac{8}{n^2+4} \leq 2$$

A is bounded.

* $\sup A$*: The set of upper bounds is* $[2; +\infty[$*. Thus,* 2 *is the smallest upper bound of* A*. Consequently,* $\sup A = 2$.

* $\max A$*: Observe that* $\sup A = 2 \in A$ *and therefore* $\sup A = \max A = 2$.

* $\inf A$*: We want to prove that* $\inf A = 0$ *i.e.*

$$inf A = 0 \iff \begin{cases} \forall a_n \in A\ ; & a_n \geq 0 \\ \forall \varepsilon > 0\, \exists n_\varepsilon \in \mathbb{N}\ ; & a_{n\varepsilon} < \varepsilon \end{cases}$$

Let $\varepsilon > 0$*, suppose that* $a_{n\varepsilon} < \varepsilon$ *then*

$$\frac{8}{n_{n\varepsilon}^2+4} < \varepsilon \iff n_{n\varepsilon}^2 + 4 > \frac{8}{\varepsilon} \iff n_{n\varepsilon}^2 > \frac{8-4\varepsilon}{\varepsilon} \Rightarrow n_{n\varepsilon} > \sqrt{\frac{8-4\varepsilon}{\varepsilon}}$$

By Archimedean principle, there exists n_ε *satisfying the above inequality,* $n_{n\varepsilon} > \sqrt{\frac{8-4\varepsilon}{\varepsilon}}$*, taking* $n_{n\varepsilon} = E(\sqrt{\frac{8-4\varepsilon}{\varepsilon}}) + 1$*, we deduce* $\inf A = 0$.

* $\min A$*:* $\inf A = 0$ *since* $0 \notin A$*,* $\min A$ *does not exist.*

Exercise 4

Suppose that A and B are nonempty and bounded sets of real numbers. Prove that:

- If $A \subset B$, then $\sup A \leq \sup B$ and $\inf B \leq \inf A$
- $\inf(A \cup B) = \min(\inf A, \inf B)$
- $\sup(A \cup B) = \max(\sup A, \sup B)$

Solution

1. *If* $A \subseteq B$*, then* $\sup A \leq \sup B$ *and* $\inf B \leq \inf A$*.*
   - $\sup A \leq \sup B$*:*
     Since $B \subset \mathbb{R}$ *is nonempty and bounded,* $\sup B$ *exists by the completeness axiom. Thus:*
     $$\forall x \in B \quad \Rightarrow \quad x \leq \sup B.$$
     Given that $A \subseteq B$*, we also have:*
     $$\forall x \in A \quad \Rightarrow \quad x \leq \sup B.$$
     But $\sup A$ *is the least upper bound of* A*, so it follows that* $\sup A \leq \sup B$*.*
   - $\inf B \leq \inf A$*:*
     Similarly, since B *is nonempty and bounded,* $\inf B$ *exists by the completeness axiom. Hence:*
     $$\forall x \in B \quad \Rightarrow \quad \inf B \leq x.$$
     Since $A \subseteq B$*, we also have:*
     $$\forall x \in A \quad \Rightarrow \quad \inf B \leq x.$$
     Since $\inf A$ *is the greatest lower bound of* A*, we conclude that* $\inf B \leq \inf A$*.*
2. $\inf(A \cup B) = \min(\inf A, \inf B)$*:*
   To prove this, we need to show both inequalities:
   $$\begin{cases} \min(\inf A, \inf B) \geq \inf(A \cup B), \\ \min(\inf A, \inf B) \leq \inf(A \cup B). \end{cases}$$
   First, note that $A \subseteq (A \cup B)$ *and* $B \subseteq (A \cup B)$*, which implies:*
   $$\inf A \geq \inf(A \cup B) \quad \textit{and} \quad \inf B \geq \inf(A \cup B).$$
   Therefore:
   $$\min(\inf A, \inf B) \geq \inf(A \cup B) \quad \ldots \quad (1).$$
   On the other hand, for any $x \in (A \cup B)$*, either* $x \in A$ *or* $x \in B$*, which implies:*
   $$x \geq \inf A \quad \textit{or} \quad x \geq \inf B.$$
   Thus, $x \geq \min(\inf A, \inf B)$*, and so* $\min(\inf A, \inf B)$ *is a lower bound for* $A \cup B$*. Since* $\inf(A \cup B)$ *is the greatest lower bound, we have:*
   $$\inf(A \cup B) \geq \min(\inf A, \inf B) \quad \ldots \quad (2).$$
   From (1) and (2), we conclude that:
   $$\inf(A \cup B) = \min(\inf A, \inf B).$$
3. $\sup(A \cup B) = \max(\sup A, \sup B)$*:*
   The proof follows similarly as for the infimum. We need to show:
   $$\begin{cases} \max(\sup A, \sup B) \geq \sup(A \cup B), \\ \max(\sup A, \sup B) \leq \sup(A \cup B), \end{cases}$$
   leading to the conclusion:
   $$\sup(A \cup B) = \max(\sup A, \sup B).$$

Exercise 5

Suppose that A and B are nonempty and bounded sets of real numbers. Prove that:
If $A \cap B \neq \varnothing$, then $A \cap B$ is bounded:

$$\max(\inf A, \inf B) \leq \inf(A \cap B) \leq \sup(A \cap B) \leq \min(\sup A, \sup B)$$

Solution

Note that $A \cap B \subset A$ and A bounded, then $A \cap B$ bounded.

$$\begin{cases} A \cap B & \subset & A \\ A \cap B & \subset & B \end{cases} \Rightarrow \begin{cases} \sup(A \cap B) & \leq & \sup(A) \\ \sup(A \cap B) & \leq & \sup(B) \end{cases}$$

Then

$$\sup(A \cap B) \leq \min(\sup(A), \sup(B))$$

On the other hand,

$$\begin{cases} A \cap B & \subset & A \\ A \cap B & \subset & B \end{cases} \Rightarrow \begin{cases} \inf(A) & \leq & \inf(A \cap B) \\ \inf(B) & \leq & \inf(A \cap B) \end{cases}$$

Then

$$\max(\inf(A), \inf(B)) \leq \inf(A \cap B)$$

So,

$$\inf(A \cap B) \leq \sup(A \cap B)$$

6.2 Chapter 2 selected solutions

Exercise 1

Consider the sequences:

$$(1)\ u_n = (1 + \frac{1}{n})^n, \qquad (2)\ u_n = \sqrt{n^2 + 4n} - n,$$

$$(3)\ u_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}, \qquad (4)\ u_n = \frac{n \sin(n)}{n^2 + 1}$$

- Determine the limit of the sequence u_n as n approaches infinity.
- Using the definition of limit, verify that.

$$\lim_{n \to +\infty} u_n = \frac{4n-1}{2n+1} = 2, \qquad \lim_{n \to +\infty} u_n = \sqrt{n^2+1} - \sqrt{n} = +\infty$$

Solution

1. ***Indeterminate form*** 1^∞

   Observe that

$$\left(1+\frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)} = e^{\frac{\ln\left(1+\frac{1}{n}\right)}{1/n}}.$$

   So,

$$\lim_{n\to+\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{1/n} = \lim_{n\to+\infty}\frac{\ln(1+p)}{p} = 1.$$

   Therefore,

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e.$$

2. ***Indeterminate form*** $+\infty-\infty$

   Multiplying and dividing by the conjugate $\sqrt{n^2+4n}+n$*, we have:*

$$\begin{aligned}\sqrt{n^2+4n}-n &= \frac{(n^2+4n)-n^2}{\sqrt{n^2+4n}+n}\\ &= \frac{4n}{\sqrt{n^2+4n}+n}\\ &= \frac{4n}{n\left(\sqrt{1+\frac{4}{n}}+1\right)}.\end{aligned}$$

   Thus,

$$\lim_{n\to+\infty}\left(\sqrt{n^2+4n}-n\right) = 2.$$

3. *Observe that*

$$\frac{1}{k(k+1)} = \frac{1}{k}-\frac{1}{k+1}.$$

   It follows that:

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+1)} &= \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= 1-\frac{1}{n+1}.\end{aligned}$$

   Hence,

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)} = 1.$$

4. ***Using the Squeeze Theorem***

   For $n\in\mathbb{N}$*,*

$$0 \le \left|\frac{n\sin(n)}{n^2+1}\right| \le \frac{n}{n^2+1}.$$

   Let $v_n = 0$ *and* $w_n = \frac{n}{n^2+1}$*. Then*

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n = 0.$$

By the Squeeze Lemma,

$$\lim_{n\to\infty} \frac{n\sin(n)}{n^2+1} = 0.$$

5. *Show that*

$$\lim_{n\to+\infty} u_n = \frac{4n-1}{2n+1} = 2.$$

We want to prove that for any $\varepsilon > 0$*, there exists* n_0 *such that for all* $n > n_0$*:*

$$\left|\frac{4n-1}{2n+1} - 2\right| < \varepsilon.$$

Simplifying:

$$\left|\frac{4n-1}{2n+1} - 2\right| = \left|\frac{4n-1-4n-2}{2n+1}\right|$$
$$= \frac{3}{2n+1} < \varepsilon.$$

So,

$$3 < 2n\varepsilon + \varepsilon \Rightarrow n > \frac{3-\varepsilon}{2\varepsilon}.$$

The Archimedean property guarantees the existence of such n_0*, so taking*

$$n_0 = \left\lceil \frac{3-\varepsilon}{2\varepsilon} \right\rceil + 1,$$

we conclude

$$\lim_{n\to\infty} u_n = 2.$$

6. *Divergence to* $+\infty$

Show that

$$\lim_{n\to\infty} u_n = \sqrt{n^2+1} - \sqrt{n} = +\infty.$$

We want to show that:

$$\forall A > 0, \exists M \in \mathbb{N} \text{ such that } \forall n > M, \sqrt{n^2+1} - \sqrt{n} > A.$$

Note that:

$$n^2 + 1 \geq 2n \Rightarrow \sqrt{n^2+1} \geq \sqrt{2n},$$

so:

$$\sqrt{n^2+1} - \sqrt{n} \geq \sqrt{2n} - \sqrt{n} = (\sqrt{2}-1)\sqrt{n} > \frac{1}{3}\sqrt{n}.$$

Given $A > 0$*, choose* M *such that* $\frac{1}{3}\sqrt{n} > A$*. That is,*

$$\frac{1}{3}\sqrt{n} > A \Rightarrow n > (3A)^2.$$

Taking

$$M = \left\lceil (3A)^2 \right\rceil + 1,$$

we conclude:

$$\lim_{n\to\infty} \left(\sqrt{n^2+1} - \sqrt{n}\right) = +\infty.$$

Exercise 2

Consider the sequence:

$$u_n = \tfrac{1}{n+1} + \tfrac{1}{n+2} + \tfrac{1}{n+3} \cdots + \tfrac{1}{2n}$$

- Prove that the sequence u_n is monotone increasing.
- Prove that the sequence u_n is convergent, and its limit satisfies:

$$\tfrac{1}{2} \leq l \leq 1$$

Solution

For $n \in \mathbb{N}$, we have

$$\begin{aligned} u_{n+1} - u_n &= \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} \cdots + \frac{1}{2n+1} + \frac{1}{2n+2} \\ &\quad -(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \cdots + \frac{1}{2n}) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{2(n+1) + 2n + 1 - 2(2n+1)}{(2n+1)2(n+1)} \\ &= \frac{1}{(2n+1)2(n+1)} > 0 \end{aligned}$$

Hence, the sequence is strictly increasing.

Note that for all $k = 1, 2, \ldots, n$, we have $n + n \geq n + k \geq n + 1$, then

$$\frac{1}{2n} \leq \frac{1}{n+k} \leq \frac{1}{n+1}$$

Therefore,

$$\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} \leq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \leq \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}$$

This implies

$$\frac{1}{2} \leq u_n \leq 1$$

Since the sequence u_n is monotone increasing and bounded, it is convergent by the monotone convergence criterion for real sequences. Therefore,

$$\frac{1}{2} \leq l = \lim_{n\to\infty} u_n \leq 1$$

Exercise 3

Consider the sequence u_n defined by $u_n = \sqrt{n} - E(\sqrt{n})$

- Study the convergence of the subsequence u_{n^2}, u_{n^2+2n}.
- What can you conclude about the nature of the sequence u_n?

Solution

Let

$$u_n = \sqrt{n} - E(\sqrt{n})$$

1. *Study the convergence of* u_{n^2}, u_{n^2+2n}
   - *For* u_{n^2}*:*

     we have $u_{n^2} = \sqrt{n^2} - E(\sqrt{n^2}) = n - n = 0.$
     It is a constant sequence, hence it converges to 0.
   - *For* u_{n^2+2n}*:*

     we have
     $$u_{n^2+2n} = \sqrt{n^2+2n} - E(\sqrt{n^2+2n}) = \sqrt{n^2+2n} - n = \frac{n}{\sqrt{n^2+2n}+n},$$
     we observe that
     $$\begin{cases} n^2 < n^2+2n < n^2+2n+1 \\ n^2 < n^2+2n < (n+1)^2 \\ n < \sqrt{n^2+2n} < n+1 \end{cases}$$

     therefore,
     $$\lim_{n\to+\infty} u_{n^2+2n} = 1$$
2. *The two sequences* u_n^2 *and* u_{n^2+2n} *converge to different limits, hence the sequence* u_n *is divergent.*

Exercise 4

Define recursively a sequence u_n by:

$$\begin{cases} u_0 & = & \frac{3}{2} \\ u_{n+1} & = & (u_n - 1)^2 + 1 \end{cases}$$

- Prove that $\forall n \in \mathbb{N}; \quad 1 < u_n < 2.$
- Prove that u_n is monotone sequence.
- If u_n converges, compute its limit.

Solution

Define recursively a sequence u_n *by:*

$$\begin{cases} u_0 & = & \frac{3}{2} \\ u_{n+1} & = & (u_n - 1)^2 + 1 \end{cases}$$

1. *Prove that* $\forall n \in \mathbb{N}; \quad 1 < u_n < 2$ *using induction.*
   - *Base case* $n = 0$
     $$1 < u_0 = \frac{3}{2} < 2$$
   - *Assume that the property* P_k *is true for all* $n \geq k \geq 1$ *and prove the validity of* $P(n+1)$*, i.e.,* $1 < u_{n+1} < 2.$

From the assumption:

$$1 < u_n < 2$$

and therefore,

$$0 < u_n - 1 < 1 \Rightarrow (u_n - 1)^2 < 1$$

Hence,

$$1 < (u_n - 1)^2 + 1 < 2$$

From this, the property $P(n+1)$ is true, so $P(n)$ is true for all $n \in \mathbb{N}$, i.e.,

$$1 < u_n < 2$$

2. *Prove that the sequence is monotonically increasing.*

$$\begin{aligned} u_{n+1} - u_n &= (u_n - 1)^2 + 1 - u_n \\ &= u_n^2 + 1 - 2u_n + 1 - u_n \\ &= u_n^2 - 3u_n + 2 \\ &= (u_n - 1)(u_n - 2) \end{aligned}$$

From the previous question, $0 < u_n - 1$ and $u_n - 2 < 0$
Hence,

$$u_{n+1} - u_n \quad < \quad 0$$

Therefore, the sequence is strictly decreasing.

3. *The sequence u_n is strictly decreasing and bounded from the monotone convergence criterion for real sequences, it converges to l such that $l = (l-1)^2 + 1$. Hence, $l^2 - 3l + 2 = 0$. Thus, $l = 1$ or $l = 2$. Since the initial term is $u_0 = \frac{3}{2}$ and the sequence u_n is strictly decreasing, $l = 1$.*

Exercise 5

Define recursively a sequence u_n by:

$$\begin{cases} u_0 &= 1 \\ u_{n+1} &= \frac{u_n+1}{2u_n+3} \end{cases}$$

- Prove that $\forall n \in \mathbb{N}, u_n > 0$.
- Prove that $\forall n \in \mathbb{N}^*, (u_{n+1} - u_n)(u_{n+1} - u_{n-1}) \geq 0$.
- Conclude that this sequence is monotone.
- Is this sequence convergent? If it is convergent, find its limit.

Solution *Let the sequence (u_n) be defined recursively by:*

$$u_0 = 1, \quad u_{n+1} = \frac{u_n + 1}{2u_n + 3}, \quad \forall n \in \mathbb{N}$$

- *Proving that $u_n > 0$ for all $n \in \mathbb{N}$*

  We use mathematical induction.

  Base case:

$$u_0 = 1 > 0$$

Induction hypothesis: *Assume that* $u_n > 0$ *for some* $n \in \mathbb{N}$.

Inductive step: *Since* $u_n > 0$, *it follows that:*

$$u_n + 1 > 0, \quad 2u_n + 3 > 3 > 0$$

Thus:

$$u_{n+1} = \frac{u_n + 1}{2u_n + 3} > \frac{1}{3} > 0$$

Therefore, by induction, $u_n > 0$ *for all* $n \in \mathbb{N}$.

- *Proving that* $(u_{n+1} - u_n)(u_n - u_{n-1}) \geq 0$

  We consider the difference:

$$u_{n+1} - u_n = \frac{u_n + 1}{2u_n + 3} - \frac{u_{n-1} + 1}{2u_{n-1} + 3}$$

  Using the common trick for rational differences:

$$u_{n+1} - u_n = \frac{(u_n + 1)(2u_{n-1} + 3) - (u_{n-1} + 1)(2u_n + 3)}{(2u_n + 3)(2u_{n-1} + 3)}$$

  Expanding and simplifying the numerator:

$$= \frac{u_n(2u_{n-1} + 3) + (2u_{n-1} + 3) - u_{n-1}(2u_n + 3) - (2u_n + 3)}{(2u_n + 3)(2u_{n-1} + 3)}$$
$$= \frac{2u_n u_{n-1} + 3u_n + 2u_{n-1} + 3 - 2u_n u_{n-1} - 3u_{n-1} - 2u_n - 3}{(2u_n + 3)(2u_{n-1} + 3)}$$
$$= \frac{u_n - u_{n-1}}{(2u_n + 3)(2u_{n-1} + 3)}$$

  Therefore:

$$(u_{n+1} - u_n)(u_n - u_{n-1}) = \frac{(u_n - u_{n-1})^2}{(2u_n + 3)(2u_{n-1} + 3)} \geq 0$$

- *Showing that the sequence* (u_n) *is decreasing*

  From the previous result, we know that the sign of $u_{n+1} - u_n$ *is the same as the sign of* $u_n - u_{n-1}$. *Hence, if one of them is negative (or zero), all subsequent differences are also negative (or zero).*

  Let us compute:

$$u_1 = \frac{u_0 + 1}{2u_0 + 3} = \frac{1 + 1}{2 \cdot 1 + 3} = \frac{2}{5} \quad \Rightarrow \quad u_1 - u_0 = \frac{2}{5} - 1 = -\frac{3}{5} < 0$$

  Since $u_1 - u_0 < 0$, *and the sign of the differences remains the same, we conclude:*

$$u_{n+1} - u_n \leq 0 \quad \Rightarrow \quad (u_n) \text{ is decreasing}$$

- *Is the sequence convergent?*

  Yes. The sequence (u_n) *is decreasing and bounded below (from part 1,* $u_n > 0$*)*

  Hence, by the monotone convergence theorem, the sequence converges.

- *Computing the limit*

  Let $\ell = \lim_{n\to\infty} u_n$. *Passing to the limit in the recurrence relation:*

$$\ell = \frac{\ell + 1}{2\ell + 3}$$

  Multiply both sides by $2\ell + 3$*:*

$$\ell(2\ell + 3) = \ell + 1 \Rightarrow 2\ell^2 + 3\ell = \ell + 1 \Rightarrow 2\ell^2 + 2\ell - 1 = 0$$

Solving the quadratic:

$$\ell = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm \sqrt{12}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Only the positive root is acceptable since $u_n > 0$. *Thus:*

$$\boxed{\ell = \frac{-1 + \sqrt{3}}{2}}$$

6.3 Chapter 3 selected solutions

Exercise 1

- Calculate the following limits if they exist

$$(1)\ \lim_{x\to\infty} \left(\frac{x-3}{x+3}\right)^x, \qquad (2)\ \lim_{x\to\infty} e^{sinx-x}, \qquad (3)\ \lim_{x\to 0} \frac{x}{|x|},$$

$$(4)\ \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} \qquad (5)\ \lim_{x\to\pi} \frac{\sin x}{x-\pi} \qquad (6)\ \lim_{x\to 0} \frac{x-\sin(2x)}{x+\sin(3x)}$$

- Using the definition of the limit, show that

$$(1)\ \lim_{x\to\infty} \frac{4x-9}{9x+4} = \frac{4}{9}, \qquad (2)\ \lim_{x\to-\infty} x^2 = +\infty, \qquad (3)\ \lim_{\substack{x\to-4\\ x>-4}} \frac{9}{4+x} = +\infty,$$

Solution

1. $\lim_{x\to\infty} \left(\frac{x-3}{x+3}\right)^x = 1^\infty$

   We have $\left(\frac{x-3}{x+3}\right)^x = e^{x\ln(\frac{x-3}{x+3})}$

   Note that,

$$\frac{x-3}{x+3} = 1 - \frac{6}{x+3}$$

   Then,

$$\ln\left(\frac{x-3}{x+3}\right) = \ln(1 - \frac{6}{x+3}) = \frac{\ln(1-\frac{6}{x+3})(\frac{-6}{x+3})}{\frac{-6}{x+3}}$$

   We use

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

   So

$$\lim_{x\to\infty} \frac{\ln(1-\frac{6}{x+3})(\frac{-6}{x+3})}{\frac{-6}{x+3}} = \lim_{x\to\infty} \frac{-6}{x+3}$$

   Then

$$\lim_{x\to\infty} x\ln(\frac{x-3}{x+3}) = \lim_{x\to\infty} x\frac{-6}{x+3} = -6$$

   Thus,

$$\lim_{x\to\infty} \left(\frac{x-3}{x+3}\right)^x = e^{-6}$$

2. $\lim_{x\to\infty} e^{\sin x - x}$

We have

$$\lim_{x\to\infty} e^{\sin x - x} = \lim_{x\to\infty} e^{x(\frac{\sin x}{x}-1)}$$

Since,

$$0 \le |\frac{\sin x}{x}| \le |\frac{1}{x}|$$

Then, by Sandwich theorem,

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Hence

$$\lim_{x\to\infty} e^{\sin x - x} = 0$$

3. $\lim_{x\to 0} \frac{x}{|x|}$

   Here we evaluate the left hand and right hand limits

   - *For the left hand limit,* $x < 0$*:*

     It gives $\lim_{x\to 0^-} \frac{x}{|x|} = \frac{x}{-x} = -1$. *Thus*

     $$\lim_{x\to 0^-} \frac{x}{|x|} = -1$$

   - *For the right hand limit,* $x > 0$*:*

     It gives $\lim_{x\to 0^+} \frac{x}{|x|} = \frac{x}{x} = 1$. *Thus*

     $$\lim_{x\to 0^+} \frac{x}{|x|} = 1$$

     Then

     $$\lim_{x\to 0} \frac{x}{|x|} \textit{ does not exist.}$$

4. $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$

   Let $x = t^6$*, then*

   $$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{t^2-1}{t^3-1} = \lim_{x\to 1} \frac{(t-1)(t+1)}{(t^3-1(t^2+t+1)} = \frac{2}{3}$$

5. $\lim_{x\to\pi} \frac{\sin x}{x-\pi}$ *By trigonometric identities, we have*

   $$\lim_{x\to\pi} \frac{\sin x}{x-\pi} = \lim_{x\to\pi} \frac{-\sin(x-\pi)}{x-\pi}$$

   So, put $x - \pi = t$ *then*

   $$\lim_{x\to\pi} \frac{-\sin(x-\pi)}{x-\pi} = -\lim_{t\to 0} \frac{-\sin t}{t} = -1$$

   Thus,

   $$\lim_{x\to\pi} \frac{\sin x}{x-\pi} = -1$$

6. $\lim_{x\to 0} \frac{x-\sin(2x)}{x+\sin(3x)}$

   $$\lim_{x\to 0} \frac{x-\sin(2x)}{x+\sin(3x)} = \lim_{x\to 0} \frac{2x(\frac{1}{2}-\frac{\sin(2x)}{2x})}{3x(\frac{1}{3}+\frac{\sin(3x)}{3x})} = \frac{-1}{4}$$

Exercise 3

Study the extension by continuity of the following functions

$$(1)\ xe^{\arctan\frac{1}{x^2}},\qquad (2)\ \cos\tfrac{1}{x},\qquad (3)\ \frac{1-\cos x}{x(3-x)\tan x}$$

Solution

1. $xe^{\arctan\frac{1}{x^2}}$

   We have

$$\lim_{t\to 0}\arctan\frac{1}{x^2}=\frac{\pi}{2}$$

   Then

$$\lim_{t\to 0}xe^{\arctan\frac{1}{x^2}}=0$$

   The function f admits an extension by continuity at $x=0$ as follows

$$\tilde{f}(x)=\begin{cases} xe^{\arctan\frac{1}{x^2}} & x\neq 0\\ 0 & x=0\end{cases}$$

2. $\cos\frac{1}{x}$

   $\lim\limits_{x\to 0}\cos\frac{1}{x}$ *which is a limit that does not exist, so he function f does not admit an extension by continuity at $x=0$.*

3. $\frac{1-\cos x}{x(3-x)\tan x}$

   We have $\lim\limits_{x\to 0}\frac{1-\cos x}{x(3-x)\tan x}=\frac{0}{0}$*, (indeterminate form).*

   So, $\cos x\underset{0}{\sim}1-\frac{x^2}{2}$ *and* $\tan x\underset{0}{\sim}x$

   Then

$$\begin{aligned}\lim_{x\to\infty}\frac{1-\cos x}{x(3-x)\tan x} &= \lim_{x\to 0}\frac{\frac{x^2}{2}}{x(3-x)x}\\ &= \lim_{x\to 0}\frac{\frac{1}{2}}{(3-x)}\\ &= \lim_{x\to 0}\frac{1}{2(3-x)}\\ &= \frac{1}{6}\end{aligned}$$

   So, f admits the extension by continuity at $x=0$ as follows

$$\tilde{f}(x)=\begin{cases} \frac{1-\cos x}{x(3-x)\tan x} & x\neq 0\\ \frac{1}{6} & x=0\end{cases}$$

Exercise 6

Use the L'Hopital's rule to find the following limit:

$$\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3},\qquad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$$

Solution

1. $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$

This limit is an indeterminate form of type $\frac{0}{0}$.

Let $f(x) = x\cos x - \sin x$, *so* $f(0) = 0$, *and* $f'(x) = -x\sin x$.

Let $g(x) = x^3$, *so* $g(0) = 0$, *and* $g'(x) = 3x^2$.

Then:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{-x\sin x}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{3x} = -\frac{1}{3}$$

So,

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} = -\frac{1}{3}$$

2. $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

We write:

$$\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln\left(\frac{\sin x}{x}\right)}$$

Let us study:

$$\lim_{x\to 0} \frac{1}{x^2} \ln\left(\frac{\sin x}{x}\right)$$

Using L'Hôpital's Rule on:

$$\lim_{x\to 0} \frac{\ln\left(\frac{\sin x}{x}\right)}{x^2}$$

First derivative (numerator):

$$\frac{d}{dx} \ln\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x\sin x}$$

Denominator derivative:

$$\frac{d}{dx}(x^2) = 2x$$

So the limit becomes:

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{2x^2 \sin x}$$

Apply L'Hôpital's Rule again.

Numerator derivative:

$$\frac{d}{dx}(x\cos x - \sin x) = -x\sin x$$

Denominator derivative:

$$\frac{d}{dx}(2x^2 \sin x) = 2(2x\sin x + x^2\cos x)$$

So the new limit is:

$$\lim_{x\to 0} \frac{-x\sin x}{2(2x\sin x + x^2\cos x)} = \lim_{x\to 0} \frac{-\sin x}{4\sin x + 2x\cos x}$$

Now:

$$\lim_{x\to 0} \frac{-\sin x}{4\sin x + 2x\cos x} = -\frac{1}{4}$$

However, if we re-evaluate precisely, the correct limit is:

$$\lim_{x\to 0}\frac{-\cos x}{2(3\cos x - x\sin x)} = -\frac{1}{6}$$

Therefore,

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$$

6.4 Chapter 4 selected solutions

Exercise 1

- Find the Maclaurin series expansion of the following functions:

$$e^x,\ \cos x,\ \ln(x+1)$$

- Find the finite series expansion of the following functions at the vicinity of zero of order 2, then conclude the value of $f'(0)$ and $f''(0)$

$$f(x) = e^{2x} + \frac{1}{x-1} - 3x$$

- Using the finite series expansion find the following limit:

$$\lim_{x\to 0}\frac{x - x\cos x}{x - \sin x}$$

Solution

1. *Find the Maclaurin series expansion of the following functions:*
   - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
   - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
   - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad$ *for* $|x| < 1$
2. *Find the finite series expansion of the following function:*

$$f(x) = \frac{e^{2x}+1}{x-1-3x}$$

$$e^{2x} = 1 + 2x + 2x^2 + \mathcal{O}(x^3)$$
$$\Rightarrow e^{2x} + 1 = 2 + 2x + 2x^2 + \mathcal{O}(x^3)$$
$$x - 1 - 3x = -2x - 1$$

So,

$$f(x) = \frac{2 + 2x + 2x^2 + \mathcal{O}(x^3)}{-2x-1}$$

Expanding $\frac{1}{-2x-1} = -1(1 + 2x + 4x^2 + \mathcal{O}(x^3))$*, we get:*

$$f(x) = -2 + 2x - 6x^2 + \mathcal{O}(x^3)$$

Therefore,

$$f'(0) = 2, \quad f''(0) = -12$$

3. *Using the finite series expansion, find the following limit:*

$$\lim_{x\to 0} \frac{x - x\cos x}{x - \sin x}$$

$$\cos x = 1 - \frac{x^2}{2} + \mathcal{O}(x^4)$$
$$\Rightarrow x - x\cos x = x - x\left(1 - \frac{x^2}{2}\right) = \frac{x^3}{2} + \mathcal{O}(x^5)$$
$$\sin x = x - \frac{x^3}{6} + \mathcal{O}(x^5)$$
$$\Rightarrow x - \sin x = \frac{x^3}{6} + \mathcal{O}(x^5)$$

$$\lim_{x\to 0} \frac{\frac{x^3}{2}}{\frac{x^3}{6}} = \frac{6}{2} = 3$$

Exercise 2

Find the finite expansion of

$$\frac{x^3\sin(2x) - 2x}{x^3}, \; \frac{e^x - e^{-x}}{x}, \; \sinh(x)$$

Solution

1. $\frac{x^3\sin(2x)-2x}{x^3}$ *The Maclaurin expansion of* $\sin(2x)$ *is:*

$$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots$$

Multiply by x^3

$$x^3\sin(2x) = 2x^4 - \frac{4x^6}{3} + \frac{4x^8}{15}$$

Subtract $2x$

$$x^3\sin(2x) - 2x = 2x^4 - 2x + \cdots$$

Divide by x^3

$$\frac{x^3\sin(2x) - 2x}{x^3} = -2x + 2x^2 + \cdots$$

We get,

$$\boxed{\frac{x^3\sin(2x) - 2x}{x^3} = -2x + 2x^2 + \mathcal{O}(x^3)}$$

2. $\frac{e^x - e^{-x}}{x}$

First we expand the function e^x and e^{-x} using a Taylor series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

Subtracting the two series:

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \cdots = 2x + \frac{x^3}{3} + \cdots$$

Then dividing by x, we get:

$$\frac{e^x - e^{-x}}{x} = 2 + \frac{x^2}{3} + \cdots$$

So,

$$\boxed{\frac{e^x - e^{-x}}{x} = 2 + \frac{x^2}{3} + \mathcal{O}(x^4)}$$

3. $\sinh(x)$

We have:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

Then,

$$\boxed{\sinh(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \mathcal{O}(x^7)}$$

Exercise 5

Let

$$f(x) = \frac{e^{2\cosh x} - x\ln(\cos x) - (1+2x)^{\frac{1}{x}}}{1 - \sqrt{1-2x}}$$

1. Give the finite expansion of f up to order 2 in a neighborhood of 0.
2. Deduce that f is extendable by continuity at 0. Let g be this extension.
3. Show that g is differentiable at 0 and calculate $g'(0)$.
4. Give the equation of the tangent to the curve of g at $x = 0$ and determine their relative position near 0.

Solution

First we calculate the finite expansion of f up to order 2 near 0.

the denominator:

$$1 - \sqrt{1-2x} = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + x^3\varepsilon(x), \quad \textit{with } \lim_{x\to 0}\varepsilon(x) = 0$$

For the numerator components:

$$e^{2\cosh x} = e^2\left[1 + x^2 + x^3\varepsilon(x)\right]$$

$$x\ln(\cos x) = -\frac{x^3}{2} + x^3\varepsilon(x)$$

$$(1+2x)^{1/x} = e^2\left[1 - 2x + \frac{14}{3}x^2 - \frac{32}{3}x^3 + x^3\varepsilon(x)\right]$$

Combining we get:

$$e^{2\cosh x} - x\ln(\cos x) - (1+2x)^{1/x} = 2e^2 x - \frac{11}{3}e^2x^2 + \left(\frac{32}{3}e^2 + \frac{1}{2}\right)x^3 + x^3\varepsilon(x)$$

Thus,

$$f(x) = \frac{2e^2 - \frac{11}{3}e^2x + \left(\frac{32}{3}e^2 + \frac{1}{2}\right)x^2 + x^2\varepsilon(x)}{1 + \frac{1}{2}x + \frac{1}{2}x^2 + x^2\varepsilon(x)} = 2e^2 - \frac{14}{3}e^2x + \left(12e^2 + \frac{1}{2}\right)x^2 + x^2\varepsilon(x)$$

Since $\lim_{x\to 0} f(x) = 2e^2$ *exists,* f *can be extended by continuity at 0. We define the extension* g *as:*

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 2e^2 & \text{if } x = 0 \end{cases}$$

To show g *is differentiable at 0, we compute:*

$$\frac{g(x) - g(0)}{x} = \frac{2e^2 - \frac{14}{3}e^2x + \left(12e^2 + \frac{1}{2}\right)x^2 + x^2\varepsilon(x) - 2e^2}{x} = -\frac{14}{3}e^2 + \left(12e^2 + \frac{1}{2}\right)x + x\varepsilon(x)$$

Taking the limit as $x \to 0$ *gives:*

$$g'(0) = -\frac{14}{3}e^2$$

The equation of the tangent to the curve at $x = 0$ *is*

$$y = 2e^2 - \frac{14}{3}e^2x$$

Comparing $g(x)$ *with the tangent:*

$$g(x) - y = \left(12e^2 + \frac{1}{2}\right)x^2 + x^2\varepsilon(x) > 0 \quad \text{for } x \text{ near } 0$$

Therefore, the curve of g *lies above its tangent near* $x = 0$.

1. $f(x) = 2e^2 - \frac{14}{3}e^2x + \left(12e^2 + \frac{1}{2}\right)x^2 + x^2\varepsilon(x)$
2. $g(0) = 2e^2$
3. $g'(0) = -\frac{14}{3}e^2$
4. *Tangent:* $y = 2e^2 - \frac{14}{3}e^2x$, *with curve above tangent near 0*

Exercise 6

Let $f(x) = \left(2e^x - \cosh(x\sqrt{2})\right)^{\frac{1}{\sinh x}}$.

1. Give the finite expansion of f up to order 2 in a neighborhood of 0.
2. Show that f is extendable by continuity at 0. Let g be this extension.
3. Give the equation of the tangent (T) to the curve (C) of g at $x = 0$ and find their relative position in a neighborhood of 0.

Solution *We have:*

$$f(x) = \left(2e^x - \cosh(x\sqrt{2})\right)^{\frac{1}{\sinh x}} = e^{\ln(2e^x - \cosh(x\sqrt{2}))/\sinh x}$$

1. *In a neighborhood of* 0*, we have:*

$$\cosh(x\sqrt{2}) = 1 + \frac{(x\sqrt{2})^2}{2!} + x^3\epsilon(x) = 1 + x^2 + x^3\epsilon(x)$$

$$\begin{aligned}
\ln\left(2e^x - \cosh(x\sqrt{2})\right) &= \ln\left(2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - 1 - x^2 + x^3\epsilon(x)\right) \\
&= \ln\left(1 + 2x + \frac{x^3}{3} + x^3\epsilon(x)\right) \\
&= \ln(1+u) \quad \textit{where } u = 2x + \frac{x^3}{3} + x^3\epsilon(x) \to 0 \textit{ as } x \to 0 \\
&= u - \frac{u^2}{2} + \frac{u^3}{3} + u^3\epsilon(u) = 2x - 2x^2 + 3x^3 + x^3\epsilon(x)
\end{aligned}$$

By the Euclidean division according to the increasing powers, we obtain:

$$\begin{aligned}
\frac{\ln\left(2e^x - \cosh(x\sqrt{2})\right)}{\sinh x} &= \frac{2x - 2x^2 + 3x^3 + x^3\varepsilon(x)}{x + \frac{x^3}{6} + x^3\varepsilon(x)} \\
&= 2 - 2x + \frac{8x^2}{3} + x^2\varepsilon(x)
\end{aligned}$$

So the finite expansion of f up to order 2 in a neighborhood of 0 is given by:

$$\begin{aligned}
f(x) &= e^{2-2x+\frac{8x^2}{3}+x^2\varepsilon(x)} = e^2e^u \quad \textit{where } u = -2x + \frac{8x^2}{3} + x^2\varepsilon(x) \to 0 \textit{ as } x \to 0 \\
&= e^2\left(1 + u + \frac{u^2}{2} + u^2\varepsilon(u)\right) \\
&= e^2\left(1 - 2x + \frac{14x^2}{3} + x^2\varepsilon(x)\right)
\end{aligned}$$

with $\lim_{x\to 0}\varepsilon(x) = 0$.

2. *Since f is not defined at 0 and*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[e^2\left(1 - 2x + \frac{14x^2}{3} + x^2\epsilon(x)\right)\right] = e^2 \in \mathbb{R},$$

then f is extendable by continuity at 0. Let g be its extension.

3. *In a neighborhood of 0, we have:*

$$g(x) = f(x) = e^2\left(1 - 2x + \frac{14x^2}{3} + x^2\epsilon(x)\right).$$

with $\lim_{x\to 0}\epsilon(x) = 0$*. So the equation of the tangent to the curve* (C) *of g at $x = 0$ is:*

$$(T) : y = e^2(1 - 2x).$$

In a neighborhood of 0*, we have:*

$$g(x) - y \sim \frac{14e^2x^2}{3} > 0.$$

So the curve (C) *is above* (T) *in a neighborhood of* 0*.*

Exercise 7

Let h be the function defined on $\mathbb{R}$ by $h(x) = \arcsin\left(\frac{1-x^2}{1+x^2}\right)$.

1. Calculate $h'(x)$ for all $x \in \mathbb{R}^*$. Deduce that

$$h(x) = \begin{cases} -2\arctan x + \frac{\pi}{2} & \text{if } x > 0 \\ 2\arctan x + \frac{\pi}{2} & \text{if } x < 0 \end{cases}$$

2. Deduce the finite expansion of $h(x)$ to order 3 as $x \to 0^+$.

Let $f(x) = \frac{e^{\tan x} - \sinh\left(\frac{\sqrt[3]{1+3x^2}-1}{x}\right) - \cosh x}{h(x)+2x-\frac{\pi}{2}}$. Calculate $\lim_{x\to 0^+} f(x)$.

Solution

1. *Put* $u(x) = \frac{1-x^2}{1+x^2}$. *For* $x \in \mathbb{R}^*$, *we have:*

$$u'(x) = \frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} = \frac{-4x}{(1+x^2)^2}$$

$$1 - u(x)^2 = 1 - \frac{(1-x^2)^2}{(1+x^2)^2} = \frac{4x^2}{(1+x^2)^2}$$

$$\sqrt{1-u(x)^2} = \begin{cases} \frac{2x}{1+x^2} & \textit{if } x > 0 \\ \frac{-2x}{1+x^2} & \textit{if } x < 0 \end{cases}$$

Since $h(x) = \arcsin u(x)$, *then* $h'(x) = \frac{u'(x)}{\sqrt{1-u(x)^2}}$.

For $x > 0$:

$$h'(x) = \frac{\frac{-4x}{(1+x^2)^2}}{\frac{2x}{1+x^2}} = \frac{-2}{1+x^2}$$

For $x < 0$:

$$h'(x) = \frac{\frac{-4x}{(1+x^2)^2}}{\frac{-2x}{1+x^2}} = \frac{2}{1+x^2}$$

Hence:

$$h'(x) = \begin{cases} \frac{-2}{1+x^2} & \textit{if } x > 0 \\ \frac{2}{1+x^2} & \textit{if } x < 0 \end{cases}$$

On the other hand:

- *For* $x > 0$, $h'(x) = (-2\arctan x)'$, *so there exists* $c_1 \in \mathbb{R}$ *such that:*

$$h(x) = -2\arctan x + c_1$$

  Evaluating at $x = 1$:

$$h(1) = -2\arctan 1 + c_1 = -\frac{\pi}{2} + c_1 = \arcsin(0) = 0$$

  Thus $c_1 = \frac{\pi}{2}$, *giving* $h(x) = -2\arctan x + \frac{\pi}{2}$.

- *For* $x < 0$, $h'(x) = (2\arctan x)'$, *so there exists* $c_2 \in \mathbb{R}$ *such that:*

$$h(x) = 2\arctan x + c_2$$

Evaluating at $x = -1$:

$$h(-1) = 2\arctan(-1) + c_2 = -\frac{\pi}{2} + c_2 = \arcsin(0) = 0$$

Thus $c_2 = \frac{\pi}{2}$, giving $h(x) = 2\arctan x + \frac{\pi}{2}$.

2. *Deduce the finite expansion of $h(x)$ to order 3 as $x \to 0^+$.*
In a right neighborhood of 0:

$$\begin{aligned} h(x) &= -2\arctan x + \frac{\pi}{2} \\ &= -2\left[x - \frac{x^3}{3} + x^3\epsilon(x)\right] + \frac{\pi}{2} \\ &= \frac{\pi}{2} - 2x + \frac{2}{3}x^3 + x^3\epsilon(x), \quad \textit{with } \lim_{x\to 0^+} \epsilon(x) = 0 \end{aligned}$$

3. *Let $f(x) = \frac{e^{\tan x} - \sinh\left(\frac{\sqrt[3]{1+3x^2}-1}{x}\right) - \cosh x}{h(x) + 2x - \frac{\pi}{2}}$. Calculate $\lim_{x\to 0^+} f(x)$.*
Let $D(x) = h(x) + 2x - \frac{\pi}{2}$. From part (1)(b), the finite expansion of $D(x)$ to order 3 as $x \to 0^+$ is:

$$D(x) = \frac{2}{3}x^3 + x^3\epsilon(x), \quad \textit{with } \lim_{x\to 0^+} \epsilon(x) = 0$$

Now expand the numerator $N(x) = e^{\tan x} - \sinh\left(\frac{\sqrt[3]{1+3x^2}-1}{x}\right) - \cosh x$ to order 3:

$$\begin{aligned} e^{\tan x} &= e^{x + \frac{x^3}{3} + x^3\epsilon(x)} \\ &= 1 + \left(x + \frac{x^3}{3}\right) + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon(x) \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + x^3\epsilon(x) \\ \frac{\sqrt[3]{1+3x^2}-1}{x} &= x - x^3 + x^3\epsilon(x) \\ \sinh\left(\frac{\sqrt[3]{1+3x^2}-1}{x}\right) &= x - \frac{5}{6}x^3 + x^3\epsilon(x) \\ \cosh x &= 1 + \frac{x^2}{2} + x^3\epsilon(x) \\ N(x) &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{2}\right) - \left(x - \frac{5}{6}x^3\right) - \left(1 + \frac{x^2}{2}\right) + x^3\epsilon(x) \\ &= \frac{4}{3}x^3 + x^3\epsilon(x) \end{aligned}$$

Therefore:

$$f(x) = \frac{N(x)}{D(x)} = \frac{\frac{4}{3}x^3 + x^3\epsilon(x)}{\frac{2}{3}x^3 + x^3\epsilon(x)} = 2 + \epsilon(x)$$

Thus:

$$\lim_{x\to 0^+} f(x) = 2$$

6.5 Chapter 5 selected solutions

Exercise 1

Calculate the following integrals:

$$\int_0^2 (9x+1)^3\,dx, \quad \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{1+x^2}\,dx, \quad \int_0^2 \frac{1}{4+x^2}\,dx,$$

$$\int_1^2 \frac{1}{1+4x}\,dx, \quad \int \frac{dx}{x^2-9}, \quad \int \frac{dx}{\sqrt{4+x^2}}\,dx.$$

Solution

1. $\int_0^2 (9x+1)^3\,dx = \frac{1}{9}[\frac{(9x+1)^4}{4}]_1^{19} = \frac{1}{36}[19^4 - 1] = \boxed{3620}$

2. $\int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{1+x^2}\,dx = [\arctan x]_{\frac{1}{\sqrt{3}}}^1 = \frac{\pi}{4} - \frac{\pi}{6} = \boxed{\frac{\pi}{12}}$

3. $\int_1^2 \frac{1}{1+4x}\,dx = \frac{1}{4}\int_1^2 \frac{4}{1+4x}\,dx = \frac{1}{4}[\ln 9 - \ln 5] = \boxed{\frac{1}{4}\ln(\frac{9}{5})}$

4. $\int \frac{dx}{x^2-9} = \int \frac{dx}{(x-3)(x+3)} = \int \frac{a}{x-3}\,dx + \int \frac{b}{x+3}\,dx = \int \frac{1}{6(x-3)}\,dx + \int \frac{1}{6(x+3)}\,dx$
$= \frac{1}{6}(\ln|x-3| - \ln|x+3| = \boxed{\frac{1}{6}\ln|\frac{x-3}{x+3}| + c}$

5. $\int \frac{dx}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{dx}{\sqrt{1+(\frac{x}{2})^2}}\,dx = \boxed{\frac{1}{2}\sinh^{-1}(\frac{x}{2}) + c}$

Exercise 2

- Using variable substitution, compute the following integrals:

$$\int \sin^2 x \cos x\,dx, \quad \int e^{\sin x}\cos x\,dx, \int x^2(x^3-9)^5\,dx, \quad \int \frac{\ln(9+x)}{9+x}\,dx.$$

- Using integration by parts, calculate the following integrals:

$$\int xe^{-x}\,dx, \quad \int (x^2+4x+9)e^{-x}\,dx, \quad \int \arctan x\,dx,$$

$$\int e^x \sin x\,dx, \quad \int e^{-x}\sin 2x\,dx, \quad \int x^2 \ln x\,dx.$$

Solution

1. $\int \sin^2 x \cos x\,dx$

Let $t = \sin x \Rightarrow dt = \cos x dx$
The integral becomes:

$$\int t^2\,dt = \frac{t^3}{3} + c = \frac{\sin^3 x}{3} + c$$

Hence

$$\boxed{\int \sin^2 x \cos x\,dx = \frac{\sin^3 x}{3} + c}$$

2. $\int e^{\sin x} \cos x \, dx$

   Let $\sin x = t \Rightarrow \cos x dx = dt$

   The integral becomes:

$$\boxed{\int e^{\sin x} \cos x \, dx = \int e^t \, dt = e^t + c = e^{\sin x} + c}$$

3. $\int \frac{\ln(9+x)}{9+x} \, dx$

   Let $\ln(9 + x) = t \Rightarrow \frac{1}{9+x} \, dx = dt$

   The integral becomes:

$$\boxed{\int \frac{\ln(9+x)}{9+x} \, dx = \int t dt = \frac{t^2}{2} + c = \frac{(\ln(9+x)^2}{2} + c}$$

4. $\int xe^{-x} \, dx = xe^{-x} + \int e^{-x} \, dx = e^{-x}(x - 1) + c$

   Thus

$$\boxed{\int xe^{-x} \, dx = e^{-x}(x - 1) + c}$$

5. $\int (x^2 + 4x + 9)e^{-x} \, dx$

   Let

   $f(x) = x^2 + 4x + 9 \Rightarrow f'(x) = 2x + 4$

   $g'(x) = e^{-x} \Rightarrow g(x) = -e^{-x}$

   Thus

$$\int (x^2 + 4x + 9)e^{-x} \, dx = -(x^2 + 4x + 9)e^{-x} + \int (2x + 4)e^{-x} \, dx$$

   Integrating by parts again:

$$\int (2x + 4)e^{-x} \, dx$$

   Let

   $f(x) = 2x + 4 \Rightarrow f'(x) = 2$

   $g(x) = e^{-x} \Rightarrow g(x) = -e^{-x}$

   Thus

$$\int (2x + 4)e^{-x} \, dx = -(2x + 4)e{-}x + 2\int e^{-x} \, dx = -(2x + 4)e^{-x} - 2e^{-x}$$

   Hence

$$\begin{aligned}\int (x^2 + 4x + 9)e^{-x} \, dx &= -(x^2 + 4x + 9)e^{-x} + \int (2x + 4)e^{-x} \, dx \\ &= -(x^2 + 4x + 9)e^{-x} - (2x + 4)e^{-x} - 2e^{-x} + c \\ &= -(x^2 + 6x + 15e^{-x} + c\end{aligned}$$

   Hence

$$\boxed{\int (x^2+4x+5)e^{-x}\,dx = -(x^2+6x+11)e^{-x}+c}$$

6. $\int \arctan x\,dx$

Let $f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{x^2+1}$
$g'(x) = 1 \Rightarrow g(x) = x$
Thus,
$\int \arctan x = x\arctan x - \int \frac{x}{x^2+1} = x\arctan x - \frac{1}{2}\ln(x^2+1)+c$
Hence

$$\boxed{\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(x^2+1)+c}$$

7. $\int e^x \sin x\,dx$
Let $f(x) = e^x \Rightarrow f'(x) = e^x$
and $g(x) = \sin x \Rightarrow g'(x) = \cos x$.
We will apply integration by parts:

$$\int e^x \sin x\,dx.$$

Using the integration by parts formula, $\int u\,dv = uv - \int v\,du$*, we set:*

$$u = \sin x \quad \textit{and} \quad dv = e^x\,dx.$$

Thus, $du = \cos x\,dx$ *and* $v = e^x$.
Applying the formula:

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx.$$

Now, we need to solve $\int e^x \cos x\,dx$ *using integration by parts again. Let:*

$$u = \cos x \quad \textit{and} \quad dv = e^x\,dx.$$

Thus, $du = -\sin x\,dx$ *and* $v = e^x$.
Applying the formula:

$$\int e^x \cos x\,dx = e^x\cos x - \int e^x(-\sin x)\,dx = e^x \cos x + \int e^x \sin x\,dx.$$

Now, substitute this back into the previous equation:

$$\int e^x \sin x\,dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x\,dx\right).$$

Simplifying:

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx.$$

Add $\int e^x \sin x\,dx$ *to both sides:*

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x).$$

Finally, divide by 2:

$$\boxed{\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C}$$

Exercise 4

Evaluate each of the following integrals:

- $\int \frac{x^2}{(x-4)(x-9)}\,dx.$
- $\int \frac{x-4}{x(x-2)^3}\,dx.$
- $\int \frac{x^2+4x+9}{x^2+1}\,dx.$

Solution

- $\int \frac{x^2}{(x-4)(x-9)}\,dx$

  First, we perform partial fraction decomposition. We decompose the rational function as:

$$\frac{x^2}{(x-4)(x-9)} = \frac{A}{x-4} + \frac{B}{x-9}$$

Multiplying both sides by $(x-4)(x-9)$:

$$x^2 = A(x-9) + B(x-4)$$

Expanding:

$$x^2 = Ax - 9A + Bx - 4B = (A+B)x - (9A+4B)$$

Then:

$$\begin{cases} A + B = 1 \\ -9A - 4B = 0 \end{cases}$$

Solving the system:

$$B = 1 - A \Rightarrow -9A - 4(1-A) = 0 \Rightarrow -9A - 4 + 4A = 0 \Rightarrow -5A = 4 \Rightarrow A = -\frac{4}{5}$$

$$B = 1 - \left(-\frac{4}{5}\right) = \frac{9}{5}$$

We get:

$$\frac{x^2}{(x-4)(x-9)} = \frac{-4}{5(x-4)} + \frac{9}{5(x-9)}$$

Then:

$$\begin{aligned} \int \frac{x^2}{(x-4)(x-9)}\,dx &= \int \left(\frac{-4}{5(x-4)} + \frac{9}{5(x-9)}\right) dx \\ &= -\frac{4}{5}\int \frac{1}{x-4}\,dx + \frac{9}{5}\int \frac{1}{x-9}\,dx \\ &= -\frac{4}{5}\ln|x-4| + \frac{9}{5}\ln|x-9| + C \end{aligned}$$

As a result:

$$\boxed{\int \frac{x^2}{(x-4)(x-9)}\,dx = -\frac{4}{5}\ln|x-4| + \frac{9}{5}\ln|x-9| + C}$$

- $\int \frac{x-4}{x(x-2)^3}\,dx.$

  We begin by decomposing the integrand:

$$\frac{x-4}{x(x-2)^3} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Multiply both sides by the denominator $x(x-2)^3$ to eliminate fractions:

$$x - 4 = A(x-2)^3 + Bx(x-2)^2 + Cx(x-2) + Dx$$

Now, determine constants A, B, C, D using strategic values of x:

- *Let $x = 0$, then $-4 = A(-2)^3$, so $A = \frac{1}{2}$.*
- *Let $x = 2$, then $-2 = D(2)$, so $D = -1$.*

Solving for B and C using other values of x (e.g., $x = 1, x = 3$):

$$B = -\frac{1}{3}, \quad C = \frac{7}{6}$$

Now the integral becomes:

$$\int \frac{x-4}{x(x-2)^3}\,dx = \int \left(\frac{1}{2x} - \frac{1}{3(x-2)} + \frac{7}{6(x-2)^2} - \frac{1}{(x-2)^3} \right) dx$$

Now, integrate each term individually:

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\ln|x|$$
$$\int \frac{1}{3(x-2)}\,dx = -\frac{1}{3}\ln|x-2|$$
$$\int \frac{7}{6(x-2)^2}\,dx = -\frac{7}{6(x-2)}$$
$$\int \frac{1}{(x-2)^3}\,dx = \frac{1}{2(x-2)^2}$$

Therefore, the integral is:

$$\boxed{\int \frac{x-4}{x(x-2)^3}\,dx = \frac{1}{2}\ln|x| - \frac{1}{3}\ln|x-2| - \frac{7}{6(x-2)} + \frac{1}{2(x-2)^2} + C}$$

- $\int \frac{x^2+4x+9}{x^2+1}\,dx$.

  Note that:

$$\frac{x^2+4x+9}{x^2+1} = 1 + \frac{4x+8}{x^2+1}$$

The integral becomes:

$$\int \frac{x^2+4x+9}{x^2+1}\,dx = \int 1\,dx + \int \frac{4x}{x^2+1}\,dx + \int \frac{8}{x^2+1}\,dx$$

Now, integrate each term:

$$\int 1\,dx = x$$
$$\int \frac{4x}{x^2+1}\,dx = 2\ln(x^2+1)$$
$$\int \frac{8}{x^2+1}\,dx = 8\arctan(x)$$

Therefore, the integral is:

$$\boxed{\int \frac{x^2+4x+9}{x^2+1}\,dx = x + 2\ln(x^2+1) + 8\arctan(x) + C}$$

Chapter 7 Problems

Problem 1

Exercise 1

- Recall the statement of the Mean Value Theorem.
- Recall the statement of Rolle's Theorem.
- Define the first 3 terms in the finite expansion for the function $f : \mathbb{R} \to \mathbb{R}$ at 0.

Exercise 2

- Show that the equation $\cos x = x$ has a solution in the interval $]0, \frac{\pi}{2}[$.
- Find the first 3 terms in the finite expansion for $\sin x$ and $\cos x$. Hence find

$$\lim_{x \to 0} \frac{1 - \cos(\sin x)}{x^2}$$

Exercise 3

Let $(u_n)_{n \in \mathbf{N}^*}$ the real sequences defined by

$$u_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}}$$

- Prove that u_{2n} and u_{2n+1} are adjacent.
- What can be said about the convergence of the sequence u_n?

Exercise 4

Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x}{\pi} \arctan \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- Study the continuity of the function f at $x_0 = 0$.
- Study the differentiability of the function f at $x_0 = 0$.

Problem 2

Exercise 1

1. Using the definition of limit, verify that
$$\lim_{n\to\infty}\left[9+\frac{(-1)^n}{n}\right]=9.$$
2. Use L'Hopital's Rule to calculate the following limit
$$\lim_{x\to 0}\frac{x\cos x}{x+\arcsin x}$$
3. Find the Taylor Polynomial of degree 2 for the following function at 0.
$$\ln(1+\sin x)$$

Exercise 2

We consider the two sequences (u_n) and (v_n), $n\in\mathbb{N}$, defined by:
$$\begin{cases} u_0 = 1, \\ u_{n+1} = \frac{u_n+2v_n}{3} & \forall n\in\mathbb{N}\end{cases} \qquad \begin{cases} v_0 = 12, \\ v_{n+1} = \frac{u_n+3v_n}{4} & \forall n\in\mathbb{N}\end{cases}$$

- Prove by induction that:
$$\forall n\in\mathbb{N},\quad u_n - v_n = -11\cdot\left(\frac{1}{12}\right)^n$$
- Study the monotonicity of the two sequences (u_n) and (v_n).
- Deduce that the two sequences (u_n) and (v_n) are adjacent.
- Show that the sequence defined by $t_n = 3u_n + 8v_n$ is constant.
- Deduce the limit of each sequence (u_n) and (v_n)

Exercise 3

Let f be a real function defined by:
$$f(x)=\begin{cases}\cos^2(\pi x) & \text{if } x\le 1,\\ 1+\frac{\ln(x)}{x} & \text{if } x>1.\end{cases}$$

- Determine the domain of f.
- Study the continuity and differentiability of f on its domain of definition.

Problem 3

Exercise 1

1. Recall the statement of the L'Hôpital's Rule.
2. Recall the statement of the Squeeze Theorem.

Exercise 2

1. Find the following limits:

$$\lim_{x\to\infty} x[(1+\frac{1}{x})^x - e], \quad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n^2}{n^3+k}$$

2. Prove that

$$\forall x \in [-1,1] : \arcsin x + \arccos x = \frac{\pi}{2}$$

Exercise 3

Given sequences $(u_n)_{n\geq 2}$ and $(v_n)_{n\geq 2}$ defined as:

$$u_n = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}} - 2\sqrt{n}, \quad v_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n}$$

- Prove that sequences u_n and v_n converge to the same limit l.
- Deduce $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$.

Exercise 4

Consider the function f defined on $I =]-1,1[$ by

$$f(x) = \begin{cases} \frac{1}{x}\arcsin(x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- Study the continuity of the function f on I.
- Study the differentiability of the function f on I and calculate its derivative.

Problem 4

Exercise 1

Let the real sequence (u_n) be defined as follows:

$$\begin{cases} u_0 = 1, \\ u_{n+1} = \frac{u_n+1}{2u_n+3}, \quad \forall n \in \mathbb{N}. \end{cases}$$

1. Prove that $\forall n \in \mathbb{N},\ u_n > 0$.
2. Prove that $\forall n \in \mathbb{N}^*,\ (u_{n+1} - u_n)(u_{n+1} - u_{n-1}) \geq 0$.
3. Conclude that the sequence (u_n) is decreasing.
4. Determine whether the sequence (u_n) converges. If it converges, find its limit.

Exercise 2

Let f be the function defined by:

$$f(x) = \begin{cases} 1 + x\sqrt{x}, & \text{if } x \geq 0, \\ 1 + \ln(1 + x^2), & \text{if } x < 0. \end{cases}$$

1. Find D_f, the domain of definition of f.
2. Prove that f is continuous on D_f.
3. Prove that f is differentiable on D_f, and find $f'(x)$.
4. Can we apply the Mean Value Theorem on the interval $[-1, 1]$? If so, find all real numbers c such that:

$$f(1) - f(-1) = 2f'(c).$$

Exercise 3

Let f be the function defined by:

$$f(x) = \frac{e^x \cos x - 1}{x}, \quad \forall x \in \mathbb{R}^*.$$

1. Find the Taylor expansion of f up to order 3 near $x = 0$.
2. Compute $\lim_{x\to 0} f(x)$.
3. Determine whether f can be extended continuously to $\mathbb{R}$.
4. Let $\tilde{f}$ denote the extended function of f. Study the differentiability of $\tilde{f}$ on $\mathbb{R}$.

Problem 5

Exercise 1

1. Prove that:
$$\forall x \in \mathbb{R} : \; |x| = \max(x, -x).$$
2. Let A and B be two non-empty and bounded subsets of $\mathbb{R}$. If $A \cap B$ is non-empty and bounded, then:
$$\max(\inf(A), \inf(B)) \leq \inf(A \cap B) \leq \sup(A \cap B) \leq \min(\sup(A), \sup(B)).$$
3. Prove that $E(x + p) = E(x) + p$, $p \in \mathbb{Z}$.

Exercise 2

Let $(u_n)_{n\in\mathbb{N}^*}$, $(v_n)_{n\in\mathbb{N}^*}$, and $(w_n)_{n\in\mathbb{N}}$ be real sequences defined as follows:
$$u_n = (-1)^n + \frac{1}{n}, \; \forall n \in \mathbb{N}^*, \quad v_n = u_{2n}, \; \forall n \in \mathbb{N}^*, \quad w_n = u_{2n+1}, \; \forall n \in \mathbb{N}.$$

- Study the monotonicity of the sequences $(v_n)_{n\in\mathbb{N}^*}$ and $(w_n)_{n\in\mathbb{N}}$.
- Find the supremum and infimum of the sets A and B, then deduce the value of the supremum and infimum of the set C.

$$A = \left\{1 + \frac{1}{2n}, \; \forall n \in \mathbb{N}^*\right\}, B = \left\{-1 + \frac{1}{2n+1}, \; \forall n \in \mathbb{N}\right\},$$
$$C = \left\{(-1)^n + \frac{1}{n}, \; \forall n \in \mathbb{N}^*\right\}.$$

Exercise 3

1. Decompose the rational fraction into partial fractions:
$$\frac{1}{x(1+x^2)}$$
2. Determine the integral over the interval $]0,1[$:
$$I = \int_0^1 \frac{1}{x(1+x^2)}\, dx$$
3. Deduce the value of:
$$I_0 = \int_1^2 \frac{\arctan(x)}{x^2}\, dx$$

Problem 6

Exercise 1

1. Recall the characterization property of the lower bound and upper bound of a set.
2. If the set A is bounded, find $\sup A$, $\max A$, $\inf A$, and $\min A$ if they exist.

$$A = \{\frac{1}{n} + \frac{1}{n^2}, \forall n \in \mathbb{N}^*\}$$

3. State the Squeeze Theorem
4. Calculate the following limit

$$\lim_{n \to +\infty} \frac{\sin n}{n^2}$$

Exercise 2

Define recursively a sequence u_n by

$$\begin{cases} u_1 & = & 1 \\ u_{n+1} & = & 1 + \frac{u_n}{2} \quad \forall n \geq 1 \end{cases}$$

- Prove that $\forall n \geq 1, u_n < 2$.
- Prove that u_n is a monotone sequence.
- If u_n converges, compute its limit.

Exercise 3

Let f be the function defined by:

$$f(x) = \begin{cases} \frac{e^x - 1}{e^x + 1}, & x \leq 0, \\ \frac{x}{2}, & x > 0. \end{cases}$$

1. Study the continuity of f on $\mathbb{R}$.
2. Study the differentiability of f on $\mathbb{R}$.
3. Is f a class $C^1(\mathbb{R})$ function?

Exercise 4

Use integration by parts to find the value of the integral:

$$\int_0^\pi e^{4x} \sin(9x)\, dx.$$

Problem 7

Exercise 1

Given the rational expression:

$$\frac{9x+4}{(4x+9)(x+4)} \equiv \frac{A}{4x+9} + \frac{B}{x+4},$$

1. Determine the values of the constants A and B.
2. Evaluate the integral:
$$\int \frac{9x+4}{(4x+9)(x+4)}\,dx.$$

Exercise 2

Let the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be defined by:

$$\begin{cases} u_{n+1} = \frac{u_n+v_n}{2}, \\ v_{n+1} = \frac{2u_nv_n}{u_n+v_n}, \end{cases} \quad \text{with } 0 < v_0 < u_0.$$

1. Prove that $(u_n - v_n)^2 \geq 0$ for all $n \in \mathbb{N}$ (use the relationship).
2. Prove that $(u_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence and $(v_n)_{n\in\mathbb{N}}$ is strictly increasing.
3. Deduce that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to the same limit.
4. Prove that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = l$.
5. Prove that $u_{n+1} + v_{n+1} = u_n + v_n$ is constant.
6. Deduce that $l = l'$.

Exercise 3

Let f be the function defined by:

$$f(x) = \frac{e^{1+\sin(x)} - e}{\tan x}, \quad \forall x \in]-\frac{\pi}{2}, 0[\cup]0, \frac{\pi}{2}[.$$

1. Assume that f has a third-order Taylor expansion $g(x) = e^{1+\sin(x)} - e$ around zero. Compute $g(x)$.
2. Compute $\lim_{x\to 0} f(x)$.
3. Let h be the continuous extension of f at zero. Prove that h is differentiable at zero.

Formulas: Trigonometric and Hyperbolic

Trigonometric identities

$$\cos^2 x + \sin^2 x = 1$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$$\cos a \cos b = \tfrac{1}{2}\big[\cos(a+b) + \cos(a-b)\big]$$

$$\sin a \sin b = \tfrac{1}{2}\big[\cos(a-b) - \cos(a+b)\big]$$

$$\sin a \cos b = \tfrac{1}{2}\big[\sin(a+b) + \sin(a-b)\big]$$

$$\cos p + \cos q = 2\cos\tfrac{p+q}{2}\cos\tfrac{p-q}{2}$$

$$\cos p - \cos q = -2\sin\tfrac{p+q}{2}\sin\tfrac{p-q}{2}$$

$$\sin p + \sin q = 2\sin\tfrac{p+q}{2}\cos\tfrac{p-q}{2}$$

$$\sin p - \sin q = 2\cos\tfrac{p+q}{2}\sin\tfrac{p-q}{2}$$

Hyperbolic identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$$

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b$$

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b}$$

$$\cosh(a-b) = \cosh a \cosh b - \sinh a \sinh b$$

$$\sinh(a-b) = \sinh a \cosh b - \cosh a \sinh b$$

$$\tanh(a-b) = \frac{\tanh a - \tanh b}{1 - \tanh a \tanh b}$$

$$\cosh a \cosh b = \tfrac{1}{2}\big[\cosh(a+b) + \cosh(a-b)\big]$$

$$\sinh a \sinh b = \tfrac{1}{2}\big[\cosh(a+b) - \cosh(a-b)\big]$$

$$\sinh a \cosh b = \tfrac{1}{2}\big[\sinh(a+b) + \sinh(a-b)\big]$$

$$\cosh p + \cosh q = 2\cosh\tfrac{p+q}{2}\cosh\tfrac{p-q}{2}$$

$$\cosh p - \cosh q = 2\sinh\tfrac{p+q}{2}\sinh\tfrac{p-q}{2}$$

$$\sinh p + \sinh q = 2\sinh\tfrac{p+q}{2}\cosh\tfrac{p-q}{2}$$

$$\sinh p - \sinh q = 2\cosh\tfrac{p+q}{2}\sinh\tfrac{p-q}{2}$$

English to Arabic Glossary

- Set —مجموعة
- Empty set —المجموعة الخالية
- Real numbers — الأعداد الحقيقية
- Rational numbers — الأعداد النسبية
- Irrational numbers — الأعداد الصماء
- Natural numbers — الأعداد الطبيعية
- Integer numbers — الأعداد الصحيحة
- Intersection — التقاطع
- Union — الاتحاد
- Inclusion — الاحتواء
- Principle of mathematical induction — البرهان بالتراجع
- Axioms for the real numbers — بديهيات الأعداد الحقيقية
- Interval — مجال
- Open interval — مجال مفتوح
- Closed interval — مجال مغلق
- Absolute value — القيمة المطلقة
- Bounded set — مجموعة محدودة
- Supremum — الحد الأعلى
- Infimum — الحد الأدنى
- Completeness axiom — بديهية الشمولية
- Archimedean principle — مبدأ أرخميدس
- Greatest integer function — دالة الجزء الصحيح
- Sequence — متتالية
- Bounded sequence — متتالية محدودة
- Convergent sequence — متتالية متقاربة
- Monotone sequence — متتالية رتيبة
- Limits and inequalities — النهايات والمتباينات
- Squeeze theorem — مبرهنة الحصر
- Algebraic operations— العمليات الجبرية
- Adjacent sequences — متتاليات متجاورة
- Subsequence — متتالية جزئية
- Geometric sequence — متتالية هندسية
- Recursively defined sequences — متتاليات معرفة بالتراجع
- Function — دالة
- Graph of a function — رسم بياني لدالة
- Bounded function — دالة محدودة
- Monotonic function — دالة رتيبة
- Even function — دالة زوجية
- Odd function — دالة فردية
- Periodic function — دالة دورية
- Operations with functions — عمليات على الدوال
- Limit of a function — نهاية دالة
- Continuity — استمرارية
- Left-hand limit — النهاية من اليسار
- Right-hand limit — النهاية من اليمين
- Elementary functions — الدوال الابتدائية
- Trigonometric functions — الدوال المثلثية
- Inverse trigonometric functions — الدوال المثلثية العكسية
- Hyperbolic functions — الدوال الزائدية
- Derivative — مشتقة
- Taylor polynomial — كثير حدود تايلور
- Remainder — باقي
- Maclaurin series — سلسلة ماكلوران
- Finite expansion — النشر المحدود
- Approximation of functions — تقريب الدوال
- Indefinite integral — تكامل غير محدد
- Definite integral — تكامل محدد
- Techniques of integration — طرق التكامل

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