

Corrigé d'intens Mécanique Analytique

2018 - 2019

- Sujet 1 -

Ex4: QCM 5pts (0,33 x 15)

- | | | |
|------|-------|-------|
| 1. F | 6. F | 11. V |
| 2. V | 7. F | 12. V |
| 3. V | 8. F | 13. V |
| 4. V | 9. V | 14. F |
| 5. F | 10. F | 15. F |

Ex3: 5pts

A/ $Q = P^\alpha q^\beta$, $P = P^\delta q^\delta$

Transformation canonique $\Rightarrow \{Q, P\} = \{P, P\} = 0$
 $\{Q, P\}_{(P,q)} = 1$

$\{Q, P\} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial P} - \frac{\partial Q}{\partial P} \frac{\partial P}{\partial q} = (\beta\delta - \alpha\delta) P^\alpha q^{\beta\delta-1}$

$\begin{cases} \alpha + \delta - 1 = 0 \\ \beta + \delta - 1 = 0 \\ \beta\delta - \alpha\delta = 1 \end{cases} \Rightarrow \begin{cases} \delta = 1 - \alpha \\ \delta = -\alpha \\ \beta = 1 + \alpha \end{cases}$

B) $H = \frac{1}{2m} \left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2 \sin^2 \theta} \right) + a(r) + \frac{b(\theta)}{r}$

1- Lagrangien: $L = \left(\sum_i \dot{q}_i P_i \right) - H$

$\dot{q}_i = \frac{\partial H}{\partial P_i}$

$\left. \begin{aligned} \dot{r} &= \frac{P_r}{m}, & P_r &= m\dot{r} \\ \dot{\theta} &= \frac{P_\theta}{mr^2}, & P_\theta &= mr^2\dot{\theta} \\ \dot{\phi} &= \frac{P_\phi}{mr^2 \sin^2 \theta}, & P_\phi &= mr^2 \sin^2 \theta \dot{\phi} \end{aligned} \right\}$

$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - a(r) - \frac{b(\theta)}{r}$

2- Eqs de Mvt: d'Euler Lagrange.

$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) = 0$

$\begin{cases} m\ddot{r} - m r \dot{\theta}^2 - m r \dot{\phi}^2 \sin^2 \theta - \frac{\partial a(r)}{\partial r} - \frac{b(\theta)}{r^2} = 0 \\ m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} - m r^2 \dot{\phi}^2 \sin \theta \cos \theta + \frac{1}{r} \frac{\partial b(\theta)}{\partial \theta} = 0 \\ \frac{d}{dt} (m r^2 \sin^2 \theta \dot{\phi}) = 0 \end{cases}$

3- Les constantes du mouvement:

$\rightarrow \frac{\partial L}{\partial t} = 0$, ϕ coordonnée cyclique et P_ϕ une impulsion généralisée conservée.

$\rightarrow \frac{dE}{dt} = \frac{dH}{dt} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

$\frac{\partial H}{\partial t} = 0$, H ne dépend pas du temps explicite. et donc E énergie Totale est une constante du Mouvement.

Ex2: Double pendule. 5pts

1) Lagrangien:

* Degré de liberté: $M = 3N - K$, $N = 2$, $K = 4$.
 $M = 2$, 2 coordonnées généralisées $\{Q_1, Q_2\}$.

* Energie cinétique et potentiel:

$T = T_1 + T_2 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$

$V = V(P_1) + V(P_2) = -(m_1 + m_2) g l_1 \cos \theta_1 - m_2 g l_2 \cos \theta_2$

$L = T - V$ on remplace $\begin{cases} m_1 = m \\ m_2 = 2m \\ l_1 = \frac{3}{2} l \\ l_2 = l \end{cases}$

2 - l'Hamiltonien: $H = \left(\sum_i \dot{q}_i p_i \right) - L$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$p_1 = (m_1 + m_2) l_1 \dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_2$$

$$p_2 = m_2 l_2 \dot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1$$

Trouver $\dot{\theta}_1 = f_1(p_1, p_2)$, $\dot{\theta}_2 = f_2(p_1, p_2)$.

$$H = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - (m_1 + m_2) l_1 g \cos \theta_1 - m_2 l_2 g \cos \theta_2.$$

On remplace $f_1(p_1, p_2)$ et $f_2(p_1, p_2)$ pour trouver $H(\theta_1, \theta_2, p_1, p_2)$.

3 - Eqs de Mouvement:

* Euler-Lagrange: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) = 0$

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 + m_2 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin \theta_1 = 0$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_2 g \sin \theta_2 = 0.$$

* Multiplicateur de Lagrange: $L' = L + \sum_j \lambda_j f_j(x, t)$

$$L' = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + m_1 g x_1 + m_2 g x_2 + \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 (x_1 + y_1 - l_1) + \lambda_4 ((x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2).$$

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{x}_i} \right) - \left(\frac{\partial L'}{\partial x_i} \right) = 0, \quad \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{y}_i} \right) - \left(\frac{\partial L'}{\partial y_i} \right) = 0$$

$$\lambda_1 = 0, \quad \lambda_4 = \frac{m_2 \ddot{x}_2 - m_2 g}{2(x_2 - x_1)} = \frac{m_2 \ddot{y}_2}{2(y_2 - y_1)}$$

$$\lambda_2 = 0, \quad \lambda_3 = \frac{m_1 \ddot{x}_1 + m_1 g + m_2 \ddot{x}_2 - m_2 g}{2x_1} = \frac{m_1 \ddot{y}_1 + m_2 \ddot{y}_2}{2y_1}$$

EX1: Spts

1 - Lagrangien

* Degré de liberté: $M = 2$ ($N = 2, K = 4$)

* Energie cinétique et Potentiel:

$$T = T_1 + T_2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = V_1 + V_2 + V_3 = \frac{1}{2} K_1 x_1^2 + \frac{1}{2} K_2 (x_2 - x_1)^2 + \frac{1}{2} K_3 x_2^2$$

$L = T - V$, on remplace m_1, m_2, K_1, K_2, K_3 .

2 - Hamiltonien: $H = \dot{x}_1 p_1 + \dot{x}_2 p_2 - L$

$$p_1 = m_1 \dot{x}_1, \quad p_2 = m_2 \dot{x}_2$$

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{4m} + K \left(\frac{3x_1^2 + x_2^2}{2} + (x_2 - x_1)^2 \right)$$

3 - Eqs de Mot:

* Euler-Lagrange: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) = 0$

$$\begin{cases} m \ddot{x}_1 + 5K x_1 - 2K x_2 = 0 \\ 2m \ddot{x}_2 + 3K x_2 - 2K x_1 = 0 \end{cases}$$

* Hamilton-Jacobi: $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{m}, \quad \dot{x}_2 = \frac{p_2}{2m}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -[3K x_1 - 2K(x_2 - x_1)]$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -[5K x_1 - 2K x_2]$$

Corrigé d'intens Mécanique Analytique

2018-2019

-Sujet 2-

Ex1: 2pts

$$L = a\dot{x}^2 + b\dot{y}^2 + c\dot{x}\dot{y} + dx^2 + ey^2.$$

$$\text{eqs de Mot: } \begin{cases} -2\ddot{x} + 3\ddot{y} = 10x \\ 3\ddot{x} - 8\ddot{y} = 14y. \end{cases}$$

Depuis les eqs d'Euler-Lagrange: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) = 0$

$$\begin{cases} 2a\ddot{x} + c\dot{y} - 2dx = 0 \\ 2b\ddot{y} + c\dot{x} - 2ey = 0 \end{cases} \rightarrow \begin{cases} a = -1, b = -4 \\ c = 3, d = 5 \\ e = 7 \end{cases}$$

Ex2: $H(q, p) = \frac{p^2}{2} - \frac{1}{2q^2}$ 4,5pts

H est une constante de mouvement parce qu'il n'est pas dépendant explicitement du temps.

$$\{q, H\} = \frac{\partial q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} = \frac{\partial H}{\partial p} = p.$$

$$\{p, H\} = -\frac{\partial H}{\partial q} = -\frac{1}{q^3}.$$

$$C(q, p, t) = \frac{pq}{2} = t H(p, q).$$

$$\{C, H\} = \left\{ \frac{pq}{2}, H \right\} - t \{H, H\} = 1.$$

$$\{C, H\} = \frac{p^2}{2} - \frac{1}{2q^3} \equiv H \text{ représente l'Hamiltonien.}$$

Eq's d'Hamilton-Jacobi:

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{1}{q^3} = \{p, H\}$$

$$\dot{q} = \frac{\partial H}{\partial p} = p = \{q, H\}$$

4) C est une constante de mouvement?

$$\frac{dC}{dt} = \{C, H\} + \frac{\partial C}{\partial t} = H + (-H) = 0.$$

Oui, C est une constante de mouvement.

Non, il faut calculer $\frac{dC}{dt}$.

Ex3: 6pts

$$V = -\frac{\alpha}{r}.$$

1) Lagrangien:

*) Degré de liberté: $M = 3$ ($N = 1, K = 0$).

*) Energie cinétique et potentiel:

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2)$$

$$V = -\frac{\alpha}{r}.$$

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2) + \frac{\alpha}{r}.$$

$$\frac{\partial L}{\partial \varphi} = 0, \varphi \text{ coordonnée cyclique}$$

p_φ une constante de mouvement.

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\partial H}{\partial t} = 0, H \text{ est une constante du mouvement.}$$

2) Desimpulsions: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$p_r = m\dot{r}, p_\theta = mr^2\dot{\theta}, p_\varphi = mr^2\sin^2\theta\dot{\varphi}.$$

3) Hamiltonien:

$$H = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\varphi}p_\varphi - L$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} - \frac{\alpha}{r}$$

4) Eq's de Mot:

$$\text{* Euler Lagrange: } \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) = 0$$

2 - l'Hamiltonien: $H = \left(\sum_i \dot{q}_i p_i \right) - L$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$p_1 = (m_1 + m_2) l_1 \dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_2$$

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Trouver $\dot{\theta}_1 = f_1(p_1, p_2)$, $\dot{\theta}_2 = f_2(p_1, p_2)$.

$$H = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \cos \theta_1 - m_2 g l_2 \cos \theta_2$$

On remplace $f_1(p_1, p_2)$ et $f_2(p_1, p_2)$ pour trouver $H(\theta_1, \theta_2, p_1, p_2)$.

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$$L' = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + m_1 g x_1 + m_2 g x_2 + \lambda_1 (x_1 - l_1) + \lambda_2 (y_1 - l_1) + \lambda_3 (x_1^2 + y_1^2 - l_1^2) + \lambda_4 ((x_2 - l_2)^2 + (y_2 - l_2)^2 - l_2^2)$$

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{x}_i} \right) - \left(\frac{\partial L'}{\partial x_i} \right) = 0, \quad \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{y}_i} \right) - \left(\frac{\partial L'}{\partial y_i} \right) = 0$$

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$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{4m} + K \left(\frac{3x_1^2 + x_2^2}{2} + (x_2 - x_1)^2 \right)$$

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$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -[3K x_1 - 2K(x_2 - x_1)]$$

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